2008 IIB 4F2, ROBUST MULTIVARIABLE CONTROL — DR J. GONCALVES, DR G-B STAN

Question 1

a) the transfer function from v_2 to v_1 is $\frac{1}{s(sM+K_d)}$. therefore

$$x = \frac{1}{s}\frac{1}{sM+K_d}(U - kx) \quad \text{or} \quad \left(1 + \frac{k}{s(sM+K_d)}\right)x = \frac{1}{s(sM+K_d)}U$$

$$\Leftrightarrow \quad (s^2M + sK_d + k)x = U$$

From Fig. 1 $\quad M\ddot{x} = -kx - K_d\dot{x} + U$. taking the Laplace transform gives $(Ms^2 + K_ds + k)X = U$.

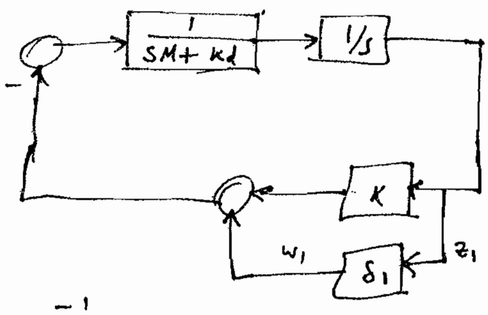
b)(i) Poles of the ~~closed loop~~ system are the roots of

$$s^2M + sK_d + k + \delta_1 = 0$$

Thus, system is stable if and only if $K_d > 0$ and $\underbrace{k+\delta_1 > 0}_{\text{feasible } \delta_i}$

or $\delta_1 > -k = -1$. So $\boxed{\delta_1 > -1}$

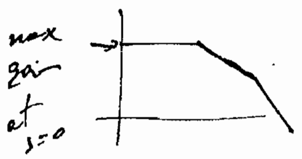
(ii)



thus,

$$\frac{z_1}{w_1} = \frac{\frac{-1}{s(sM+K_d)}}{1 + \frac{k}{s(sM+K_d)}} = \frac{-1}{s^2M + sK_d + k}$$

$$\left\|\frac{z_1}{w_1}\right\|_\infty = \left\|\frac{-1}{s^2\frac{1}{10} + s\frac{11}{10} + 1}\right\|_\infty = \left\|\frac{-10}{s^2+11s+10}\right\|_\infty = \left\|\frac{-10}{(s+1)(s+10)}\right\|_\infty$$

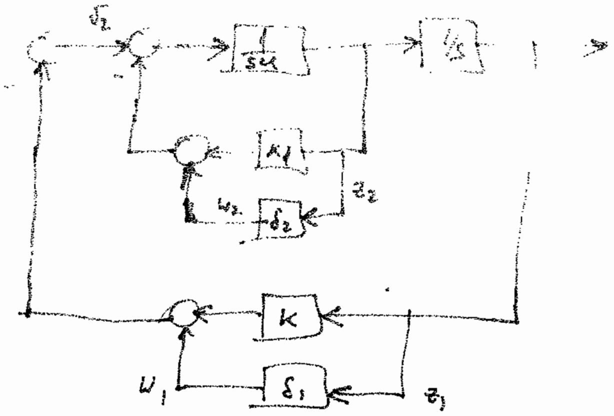


$$= 1$$

$\Rightarrow \quad |\delta_1| < 1$ by the small gain theorem

the answer in Part b (i) is precise (nec. and suf.). the small gain theorem is only sufficient here since δ_1 is a real number ~~(with this structure)~~ and ~~so~~ not an arbitrary system.

(iii)



$$\begin{cases} z_1 = \frac{1}{s} z_2 \\ z_2 = \frac{1}{sM}\left(v_2 - K_d z_2 - w_2\right) \\ v_2 = -\frac{k}{s} z_2 - w_1 \end{cases} \Leftrightarrow \left(1 + \frac{K_d}{sM} + \frac{k}{s^2 M}\right) z_2 = -\frac{1}{sM}(w_1 + w_2)$$

$$\Leftrightarrow \quad z_2 = \frac{-s}{s^2 M + K_d s + k}(w_1 + w_2)$$

So, $$G = \frac{-1}{s^2 M + sK_d + k}\begin{bmatrix} 1 & 1 \\ s & s \end{bmatrix}$$

(iv) we start by finding $\bar{\sigma}\left(\begin{bmatrix} 1 & 1 \\ j\omega & j\omega \end{bmatrix}\right) = \lambda_{max}\left(\begin{bmatrix} 1 & -j\omega \\ 1 & -j\omega \end{bmatrix}\begin{bmatrix} 1 & 1 \\ j\omega & j\omega \end{bmatrix}\right)$

$$= \sqrt{1+\omega^2}\, \lambda_{max}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \sqrt{2}\sqrt{1+\omega^2}$$

then $$\bar{\sigma}(G(j\omega)) = \frac{10}{\cancel{\sqrt{1+\omega^2}}\sqrt{\omega^2+100}} \sqrt{2} \cdot \cancel{\sqrt{1+\omega^2}} = \frac{\sqrt{2}\, 10}{\sqrt{\omega^2+100}}$$

$$\Rightarrow \sup_{\omega} \bar{\sigma}(G(j\omega)) \underset{\omega=0}{=} \sqrt{2} \Rightarrow \mu(G(j\omega)) \leq \sqrt{2}$$

(v) $\mu(G(j\omega)) = 0$ $\forall \omega$ since no Δ makes $I - G\Delta$ singular. This follows from the fact that the polynomial $Ms^2 + \underbrace{(K_d + \delta_2)}_{>0}s + \underbrace{k + \delta_1}_{>0} = 0$ has all its roots on the LHP for any δ_1, δ_2 s.t. $k + \delta_1 > 0$ and $K_d + \delta_2 > 0$.

Question 2

(a) CARE: $XA + A^TX + C_1^TC_1 - XB_2B_2^TX = 0$ (*)

with $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_1 = \begin{bmatrix} 2 & 0 \end{bmatrix}$

$$(*) \Leftrightarrow \begin{bmatrix} 4 - X_2^2 & X_1 - X_2X_3 \\ X_1 - X_2X_3 & 2X_2 - X_3^2 \end{bmatrix} = 0$$

$\Rightarrow X_2 = \pm 2$

$\Rightarrow X_3 = \pm\sqrt{\pm 4}$ → we have to chose $X_2 = +2$ for X_3 to be real

$\Rightarrow X_1 = \pm 4$

So, either $X_a = \begin{bmatrix} +4 & 2 \\ 2 & 2 \end{bmatrix}$ or $X_b = \begin{bmatrix} -4 & 2 \\ 2 & -2 \end{bmatrix}$

We need $(A - B_2B_2^TX)$ stable

For $X = X_a$, we have $A - B_2B_2^TX_a = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$

which is stable since its eigenvalues are given by $\lambda^2 + 2\lambda + 2 = 0$ (stable by Routh-Hurwitz)

(b) The stabilizing optimal controller is given by

$u(t) = -B_2^T X_a x(t)$

$\Rightarrow \boxed{u(t) = -2x_1(t) - 2x_2(t)}$

$$\min_{K(s) \text{ stabilizing}} \| F_\ell(P(s), K(s)) \|_2 = \sqrt{2\pi}\sqrt{\text{trace}(B_1^T X B_1)}$$

$$= \sqrt{2\pi}\sqrt{X_{a1}} = 2\sqrt{2\pi}$$

(c) (i) $\mathcal{F}_\ell(P(s), K(s))\Big|_{u=[k_1 \;\; k_2]x}$

$u = [k_1 \;\; k_2]\, x \Rightarrow$ the closed-loop system state-space realization is given by

$$\begin{cases} \dot{x} = \overbrace{\begin{bmatrix} 0 & 1 \\ k_1 & k_2 \end{bmatrix}}^{\tilde{A}} x + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{\tilde{B}} w \\ z = \overbrace{\begin{bmatrix} 2 & 0 \\ k_1 & k_2 \end{bmatrix}}^{\tilde{C}} x \\ y = x \end{cases}$$

$$\mathcal{F}_\ell(P(s), K(s)) = \overbrace{\begin{bmatrix} 2 & 0 \\ k_1 & k_2 \end{bmatrix}}^{\tilde{C}} \overbrace{\begin{bmatrix} s & -1 \\ -k_1 & s-k_2 \end{bmatrix}^{-1}}^{(sI-\tilde{A})^{-1}} \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{\tilde{B}}$$

$$= \frac{1}{\underbrace{s^2 - k_2 s - k_1}} \begin{bmatrix} \cdots \\ \cdots \end{bmatrix}$$

stable $\forall\, k_1 < 0,\; k_2 < 0$ by Routh-Hurwitz

(ii) observability grammian L satisfies the equation

$$\tilde{A}^T L + L\tilde{A} + \tilde{C}^T\tilde{C} = 0 \;\; (*)$$

$$L = \begin{pmatrix} \ell_1 & \ell_2 \\ \ell_2 & \ell_3 \end{pmatrix} \Rightarrow (*) \text{ writes } 0 = \begin{bmatrix} 2k_1\ell_2 + (4+k_1^2) & \ell_1 + k_2\ell_2 + k_1\ell_3 + k_1k_2 \\ \ell_1 + k_2\ell_2 + k_1\ell_3 + k_1k_2 & 2(\ell_2 + k_2\ell_3) + k_2^2 \end{bmatrix}$$

which leads to $\ell_2 = -\dfrac{k_1^2+4}{2k_1}$

$$\ell_3 = \frac{k_1^2+4}{2k_1k_2} - \frac{k_2}{2}$$

$$\ell_1 = \frac{-k_1(k_1^2+4) + 4k_2^2}{2k_1k_2}$$

(iii) $\|\mathcal{F}_\ell(P(s), K(s))\|_2 = \sqrt{2\pi}\sqrt{\operatorname{trace}(\tilde{B}^T L \tilde{B})}$

$$= \sqrt{2\pi}\sqrt{\ell_1} = \sqrt{2\pi}\sqrt{\frac{-k_1(k_1^2+4)+4k_2^2}{2k_1k_2}}$$

(iv) In (b) we saw that $u^* = -2x_1 - 2x_2$

So $k_1^* = -2$ and $k_2^* = -2$ for wich the min $\|\mathcal{F}_\ell\|_2$ is achieved.

Question 3

(a) $V(x,t) = \min_{u \in U} \left(\int_t^T r(x(t), u(t))\,dt + J_T(x(T)) \right)$

$V(x,t)$ is the optimal additional cost from time t onwards, if the state at time "t" is "x".

By definition, we have:

$$V(x,T) = J_T(x(T))$$

$$V(x,0) = J^*(x) = \text{optimal cost for the original problem with } x_0 = x(0) = x_0$$

(b) The dynamic programming eqn satisfied by $V(x,t)$ is a partial differential equation, known as the Hamilton–Jacobi–Bellman PD equation:

$$\text{HJB}: \quad -\frac{\partial V}{\partial t}(x,t) = \min_{u \in U} \left(r(x,u) + \frac{\partial V}{\partial x}(x,t)\, f(x,u) \right)$$

It captures the "principle of optimality":
If we ~~know~~ the optimal cost from time "~~t+dt~~" onwards (also called the cost-to-go from time "~~t+dt~~"), then the cost-to-go from time "t" is the minimum over the current control input ($u(t)$) of the sum of the incremental cost associated with that input and the cost-to-go from where we end up.

It simplifies the problem ~~by transforming the~~ of optimization over $u(\cdot)$ as a function of time by transforming it into a problem of pointwise optimization over $u \in U$.

(c) (i) HJB: $-\dfrac{\partial V}{\partial t} = \min_u \left(u^2 + \dfrac{\partial V}{\partial x}(x+u) \right)$

$$V(x(T),T) = (x(T))^2$$

The incremental cost and terminal cost being quadratic, this problem belongs to the class of Continuous-Time LQR problems, for which $V(x,t) = x(t)\, X(t)\, x(t)$ with $X(t) = X^T(t) \geq 0$

Solution to this class of problems is given by solving the ODE : $-\dot{X} = Q + XA + A^T X - XBR^{-1}B^T X$ ⊛

with the terminal condition $X(T) = 1$

Here · $A = 1$, $B = 1$, $Q = 0$, $R = 1$, $X(T) = 1$

So, ⊛ writes $-\dot{X} = 2X - X^2$, $X \in \mathbb{R}$

which corresponds to the Bernoulli differential equation with $k = 2$, $c = -2$, $d = 1$, $x(t) = X(t)$

(ii) $\dot{X} = -2X + X^2$, $X(T) = 1$

Pose $\begin{cases} z = X^{-1} \Rightarrow & \dot{z} = -X^{-1}\dot{X}X^{-1} \\ X = z^{-1} \Rightarrow & \dot{X} = -z^{-1}\dot{z}\,z^{-1} \end{cases}$

We obtain

$$-z^{-1}\dot{z}\,z^{-1} = -2z^{-1} + z^{-2}$$

$$\Leftrightarrow \boxed{\dot{z} = +2z - 1} \quad \text{with} \quad z(T) = \frac{1}{X(T)} = 1$$

which is a linear ODE

The solution is of the form $z(t) = \alpha e^{2t} + \beta$

Inserting this solution into the eqn gives

$$\cancel{2\alpha e^{2t}} = +\cancel{2\alpha e^{2t}} + 2\beta - 1$$

$$\Rightarrow \beta = \frac{1}{2}$$

$$z(T) = 1 = \alpha e^{2T} + \frac{1}{2} \Rightarrow \alpha = \frac{1}{2} e^{-2T}$$

$$\Rightarrow z(t) = \frac{1}{2}\left(e^{2(t-T)} + 1\right)$$

$$\Rightarrow \boxed{X(t) = \frac{1}{z(t)} = \frac{2}{1 + e^{2(t-T)}}}$$

Optimal cost $= V(x_0, 0) = x_0^2 X(0) = x_0^2 \dfrac{2e^{2T}}{1 + e^{2T}}$

Optimal control $= u^*(t) = -X(t)x(t) = -\dfrac{2}{1 + e^{2(t-T)}} x(t)$

$$V(2,0)\Big|_{T=10} = 4\,\frac{2e^{20}}{1 + e^{20}}$$

$$u(t) = -\frac{2}{1 + e^{2(t-10)}} x(t)$$