2008 IIB 4F3 NONLINEAR AND PREDICTIVE CONTROL PROF JM MACIEJOWSKI

Module 4F3: Nonlinear and Predictive Control
Solutions 2008

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13 May 2008

1. (a) Bookwork. Answers should include:
   - Assume limit cycle exists; analysis using Fourier series.
   - Low-pass assumption, hence examine first harmonics only.
   - Harmonic balance condition.
   - Graphical intersection test for condition $G(j\omega) = -1/N(E)$.
   - Approximate amplitude and frequency predictions.
   - Stability/instability of limit cycle predicted.

   (b) Let the input to the nonlinearity be $e(t) = E\sin(\omega t)$, and let $N(E)$ denote the describing function. Since the nonlinear characteristic shown is an odd function, $N(E)$ is real, because no phase shift occurs through the nonlinearity.
   Thus $N(E) = U_1/E$, where U_1 is given by

$$U_1 = \frac{1}{\pi}\int_0^{2\pi} f(E\sin\theta)\sin\theta d\theta = \frac{2}{\pi}\int_0^{\pi} f(E\sin\theta)\sin\theta d\theta \qquad (1)$$

since $f(.)$ is an odd function. But, from Fig.1 of the exam paper, $0 \le f(e)/e \le 0.5$ for $e \neq 0$. Hence

$$\begin{aligned}
\frac{U_1}{E} &= \frac{2}{\pi}\int_0^{\pi} \frac{f(E\sin\theta)}{E}\sin\theta d\theta && (2)\\
&= \frac{2}{\pi}\int_0^{\pi} \frac{f(E\sin\theta)}{E\sin\theta}\sin^2\theta d\theta && (3)\\
&\le \frac{2}{\pi}\int_0^{\pi} 0.5\sin^2\theta d\theta && (4)\\
&= \frac{2}{\pi}\int_0^{\pi/2} \sin^2\theta d\theta = \frac{2}{\pi}\times\frac{\pi}{4} = \frac{1}{2} && (5)
\end{aligned}$$

Similarly, $U_1/E \ge 0$. Thus $\boxed{0 \le N(E) \le 0.5}$ as required.

   (c) The describing function method predicts a limit cycle if there is an intersection between the graphs of $-1/N(E)$ and $G(j\omega) = a/j\omega(j\omega+1)^2$.
   Since $0 \le N(E) < 1/2$, we must have $-\infty < -1/N(E) < -2$, so a limit cycle will be predicted only if $a/j\omega_0(j\omega_0+1)^2 < -2$ if the Nyquist locus crosses the real axis at frequency ω_0. Now we show that this does not happen for $a = 1$, in two (alternative) ways:
   Method 1: Find exactly where the Nyquist locus crosses the real axis:

$$\Im\left(\frac{1}{j\omega_0(j\omega_0+1)^2}\right) = 0 \Rightarrow \arg((j\omega_0+1)^2) = \frac{\pi}{2} \Rightarrow \arg(j\omega_0+1) = \frac{\pi}{4} \Rightarrow \omega_0 = 1. \qquad (6)$$

Hence

$$\left|\frac{1}{j\omega_0(j\omega_0+1)^2}\right| = \left|\frac{1}{j(j+1)^2}\right| = \frac{1}{2}. \tag{7}$$

Clearly there is no intersection between the Nyquist locus and $-1/N(E)$.
Method 2: Show that $\Re\{G(j\omega)\} > -2$ for all ω:

$$\Re\left\{\frac{1}{j\omega(1+j\omega)^2}\right\} = \Re\left\{\frac{-j\omega(1-j\omega)^2}{\omega^2(1+\omega^2)^2}\right\} \tag{8}$$

$$= \Re\left\{\frac{-j\omega(1-2j\omega-\omega^2)}{\omega^2(1+\omega^2)^2}\right\} = \frac{-2\omega^2}{\omega^2(1+\omega^2)^2} = \frac{-2}{(1+\omega^2)^2} \tag{9}$$

So the whole Nyquist plot lies to the right of the vertical line $\Re\{z\} = -2$, and hence cannot intersect the graph of $-1/N(E)$.
Figure 1 shows the Nyquist plot and the graph of $-1/N(E)$.

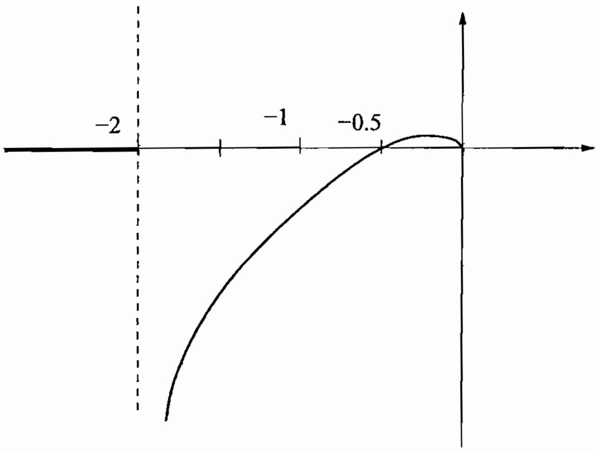

Figure 1: Nyquist locus and graph of $-1/N(E)$, with $a = 1$, for Q.1(c). Note that the graph of $-1/N(E)$ may stop some way to the left of -2.

(d) The nonlinearity shown in Fig.1 of the exam paper lies in the sector $(0, 1/2)$ — see Fig.2. The interior of the 'circle' of the circle criterion thus becomes the half-plane to the left of the vertical line through -2. The linear system is stable, so the circle criterion requires 0 encirclements of the 'circle' by the Nyquist locus, *ie* the locus should lie to the right of the vertical line. That this is the case can be established as in part (c) 'Method 2'.

Hence the conditions of the Circle Criterion are satisfied, and the feedback loop is globally asymptotically stable.

(e) Now suppose that $a = 3$. This 'expands' the Nyquist locus by a factor of 3 (ie the modulus is increased by a factor of 3 at each point, while the argument is unchanged). The Circle criterion is no longer satisfied (see Fig.3). Thus stability of the system can no longer be deduced.

Many candidates said that the Circle criterion showed that the system is unstable with $a = 3$, which is not correct, because the Circle criterion gives only a sufficient condition for stability, not a necessary condition.

The point at which the Nyquist locus crosses the negative real axis is now changed from $-1/2$ to $-3/2$. This is still to the right of -2, so still does not intersect the graph of $-1/N(E)$. Thus the describing function continues to predict that no limit cycle exists.

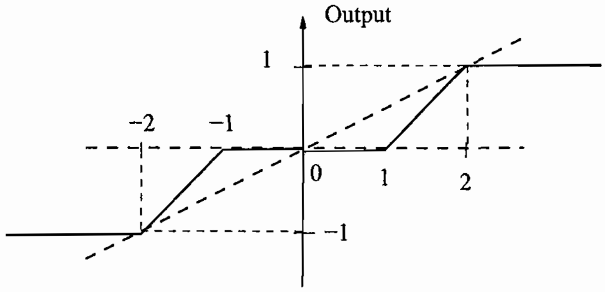

Figure 2: Showing that the nonlinearity lies in the sector $(0, 0.5)$, for Q.1(d)

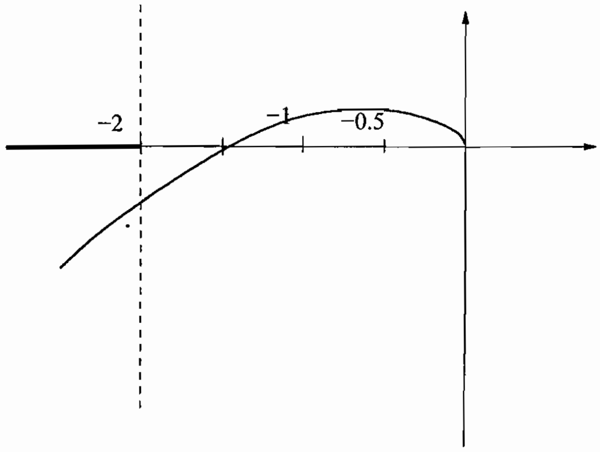

Figure 3: Nyquist locus and graph of $-1/N(E)$, with $a = 3$, for Q.1(e).

2. (a) Assume the dynamical system is defined by $\dot{x} = f(x)$.

Direct method: Look for a Lyapunov function, ie a function $V(x)$ which satisfies:

$$V(x) > 0 \quad \text{if} \quad x \neq 0 \tag{10}$$

$$V(x) \quad \text{continuous} \tag{11}$$

$$\dot{V}(x) \leq 0 \tag{12}$$

in some neighbourhood of an equilibrium, where $\dot{V}(x) = \nabla V^T f(x)$. If such a function is found then the equilibrium is stable.

If the third condition can be strengthened to $\dot{V}(x) < 0$ then the equilibrium is asymptotically stable.

If the third condition cannot be strengthened, then asymptotic stability might still be proved using LaSalle's Theorem, by showing that $\dot{V}(x)$ cannot remain at zero on any non-trivial trajectory.

Indirect method: Linearise the system in the neighbourhood of an equilibrium, to obtain $\dot{x} = Ax$, where $A = \partial f/\partial x$, evaluated at the equilibrium. Find the eigenvalues $\{\lambda_i\}$ of A. If $\Re\{\lambda\} < 0$ for each i then the equilibrium is asymptotically stable. If $\Re\{\lambda\} > 0$ for any i then the equilibrium is unstable. If $\Re\{\lambda\} = 0$ for some i and $\Re\{\lambda\} \leq 0$ for all i then stability of the equilibrium cannot be determined.

(b) i. For an equilibrium we need $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$. Thus from the first equation we get $x_1 = x_2$ at any equilibrium. Then the second equation requires that

$$-x_1 + (ax_1 + bx_1)^2 = 0 \tag{13}$$

namely

$$x_1 = 0 \quad \text{or} \quad x_1 = \pm\frac{1}{a+b} \tag{14}$$

Thus the three equilibria are:

$$x_1 = 0, \qquad x_2 = 0 \tag{15}$$

$$x_1 = \frac{1}{a+b}, \quad x_2 = \frac{1}{a+b} \tag{16}$$

$$x_1 = -\frac{1}{a+b}, \quad x_2 = -\frac{1}{a+b} \tag{17}$$

ii. We have:

$$\frac{\partial f_1}{\partial x_1} = -1, \qquad \frac{\partial f_1}{\partial x_2} = +1 \tag{18}$$

$$\frac{\partial f_2}{\partial x_1} = -1 + 2(ax_1 + bx_2)ax_2, \quad \frac{\partial f_2}{\partial x_2} = 2(ax_1 + bx_2)bx_2 + (ax_1 + bx_2)^2 \tag{19}$$

So the three linearised systems are:
At $x_1 = x_2 = 0$:

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} x \tag{20}$$

At $x_1 = x_2 = 1/(a+b)$, note that $ax_1 + bx_2 = 1$, and hence:

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ -1 + \frac{2a}{a+b} & \frac{2b}{a+b} + 1 \end{bmatrix} x \tag{21}$$

At $x_1 = x_2 = -1/(a+b)$, note that $ax_1 + bx_2 = -1$, and hence:

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ -1 + \frac{2a}{a+b} & \frac{2b}{a+b} + 1 \end{bmatrix} x \tag{22}$$

so that the linearisations at $x_1 = x_2 = 1/(a+b)$ and at $x_1 = x_2 = -1/(a+b)$ are the same.

iii. Now check stability at each equilibrium by examining the eigenvalues of the linearised system. Note that a quadratic polynomial has all roots in the left half-plane if and only if all its coefficients have the same sign (from the formula for solving a quadratic equation, or from the Routh-Hurwitz criterion).
At $x_1 = x_2 = 0$: The eigenvalues are the solutions of

$$\begin{vmatrix} \lambda + 1 & -1 \\ 1 & \lambda \end{vmatrix} = 0 \tag{23}$$

namely

$$\lambda^2 + \lambda + 1 = 0 \tag{24}$$

which has both roots with $\Re(\lambda) < 0$. Thus, by Lyapunov's indirect method, the equilibrium at $(0,0)$ is stable.
At $x_1 = x_2 = \pm 1/(a+b)$: The eigenvalues are the solutions of

$$\begin{vmatrix} \lambda + 1 & -1 \\ 1 - \frac{2a}{a+b} & \lambda - 1 - \frac{2b}{a+b} \end{vmatrix} = 0 \tag{25}$$

namely

$$\lambda^2 - \frac{2b}{a+b}\lambda - \left(1 + \frac{2b}{a+b}\right) + \left(1 - \frac{2a}{a+b}\right) = 0 \tag{26}$$

or

$$\lambda^2 - \frac{2b}{a+b}\lambda - 2 = 0 \tag{27}$$

Again the coefficients do not all have the same sign, so at least one of the eigenvalues must have positive real part. Thus the equilibria at $x_1 = x_2 = \pm 1/(a+b)$ are unstable.

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3. (a) Advantages of predictive control:

- Respects input and state (output) constraints. Enables operation closer to constraints, because of nonlinear action. This leads to lower costs/higher profits.
- Principle of operation easily understood by plant operators.
- Naturally handles multivariable control problems.
- Changes of plant behaviour easily accommodated by updating internal model.

Disadvantages of predictive control:

- Computational complexity: requires optimisation problem to be solved on-line. (Or storage of pre-computed control law in extremely large database — this answer not really expected.)
- Needs good model if constraint satisfaction is to be observed accurately.
- Lack of 'transparency' of operation; difficult to be sure about result of optimisation algorithm; difficult to certify or qualify for safety-critical tasks.
- Versions which guarantee robustness properties give rise to even greater computational complexity than the basic version — currently not implementable for many applications.

I expect the 'illustrations' to be drawn from the paper-making examples presented in Dr Austin's lecture. But reference to any other appropriate control problem is acceptable (eg one of those discussed in either of the recommended books).

With one exception, the illustrations were rather perfunctory.

(b) i. Suppose a (continuous) control law $u_k = \kappa(x_k)$ is applied, such that $\kappa(x_e) = u_e$ and $Ax_e + Bu_e = x_e$ (so that x_e is an equilibrium of the closed-loop system). If there exists a continuous function V defined on some region S containing x_e in its interior, such that

- $V(x_e) = 0$
- $V(x) > 0$ for all $x \in S$ if $x \neq x_e$
- $V(Ax + B\kappa(x)) - V(x) \leq 0$ for all $x \in S$

then V is a control Lyapunov function. (And x_e is a stable equilibrium, but this is not asked for here.)

ii. A terminal control law is a control law $u_k = \kappa(x_k)$ that is assumed to be applied to the system after the end of the prediction horizon (ie for $k \geq N$). (If constraints are present, it is usually assumed that all the constraints will be inactive for $k \geq N$.)

iii. Bookwork, as follows: The strategy here is to find a condition under which the value function is a control Lyapunov function.

Let U denote the sequence of controls $(u_0, \ldots, u_{N-1})$, and let $U^*(x) = (u_0^*, \ldots, u_{N-1}^*)$ denote the optimal sequence:

$$U^*(x) = \arg\min_U J(x, U) \tag{28}$$

Let $V^*(x)$ denote the value function: $V^*(x) = J(x, U^*)$, and let $(x_1^*, \ldots, x_N^*)$ denote the predicted state sequence under the assumption that the control sequuence U^* will be applied.

Note that $(x = 0, u = 0)$ is indeed an equilibrium of the closed-loop system, because $U^*(0) = 0$ and $V^*(0) = 0$.

Clearly $V^*(x_0) > 0$ if $x_0 \neq 0$, since $J(x_0, U) \geq x_0^T Q x_0 > 0$.

If the control u_0^* is applied at the first instant, a possible control sequence at the next instant is $(u_1, \ldots, u_{N-1}, Kx_N^*)$. This will give the cost

$$J(Ax_0 + Bu_0^*, (u_1, \ldots, u_{N-1}, Kx_N^*)) = x_{N+1}^T P x_{N+1} + \sum_{k=1}^{N} (x_k^{*T} Q x_k^* + u_k^{*T} R u_k^*)$$

$$= V^*(x_0) + x_{N+1}^T P x_{N+1} - x_0^T Q x_0 - u_0^{*T} R u_0^* - x_N^T P x_N + x_N^T Q x_N + x_N^{*T} K^T R K x_N^* \quad (29)$$

(by subtracting the terms which have come out from $V^*(x_0)$ and adding new terms entering in the next step). But $x_{N+1} = Ax_N^* + Bu_N^* = (A+BK)x_N^*$. Hence

$$J(Ax_0 + Bu_0^*, (u_1, \ldots, u_{N-1}, Kx_N^*)) =$$
$$V^*(x_0) + x_N^{*T}(A+BK)^T P (A+BK) x_N^* - x_0^T Q x_0 - u_0^{*T} R u_0^* - x_N^T P x_N + x_N^T Q x_N + x_N^{*T} K^T R K x_N^*$$
$$\leq V^*(x_0) \quad (30)$$

if

$$\mathbf{(A+BK)^T P (A+BK) - P + K^T R K + Q \leq 0} \quad (31)$$

This is the condition that guarantees closed-loop stability of the origin, because

$$V^*(Ax_0 + Bu_0^*) \leq J(Ax_0 + Bu_0^*, (u_1, \ldots, u_{N-1}, Kx_N^*)) \quad (32)$$

since V^* is the optimised value of J, hence the condition (31) ensures that

$$V^*(Ax_0 + Bu_0^*) \leq V^*(x_0) \quad (33)$$

and hence that $V^*(.)$ is a control Lyapunov function.
Note: A simplified derivation for the case $K = 0$ would be given some credit, but note that the condition then becomes $A^T P A - P \leq -Q$, which can only be satisfied if the system is open-loop stable.

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4. (a) A quadratic programming optimisation problem is a problem of the form:

$$\min_{\theta} \frac{1}{2} \theta^T \Omega \theta + g^T \theta \qquad (\theta \in \mathbb{R}^n, g \in \mathbb{R}^n, \Omega \in \mathbb{R}^{n \times n}) \quad (34)$$

subject to

$$M\theta \leq m \quad \text{and} \quad H\theta = h \quad (35)$$

for some matrices M, H and some vectors m, h.
(The factor 1/2 is just a convention and can be omitted. The condition $\Omega \geq 0$ is sometimes included in the definition — a *convex* QP is then obtained.)

Surprisingly many solutions omitted the constraints, thus defining a simple least-squares rather than a QP problem.

(b) Let x_0 be the measurement of the current state vector, x_1, x_2 its predicted values at the next two steps, and u_0, u_1 the inputs to be applied at the next two steps. Then

$$x_1 = Ax_0 + Bu_0 = Ax_0 + [B,\ 0] \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \quad (36)$$

$$x_2 = Ax_1 + Bu_1 = A^2 x_0 + ABu_0 + Bu_1 = A^2 x_0 + [AB,\ B] \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \quad (37)$$

Thus the cost to be minimised is:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} =$$

$$\left(\begin{bmatrix} A \\ A^2 \end{bmatrix} x_0 + \begin{bmatrix} B & 0 \\ AB & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right)^T \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \left(\begin{bmatrix} A \\ A^2 \end{bmatrix} x_0 + \begin{bmatrix} B & 0 \\ AB & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \right) +$$

$$\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \quad (38)$$

which is a quadratic function of the decision variables u_0, u_1. The quadratic term in x_0 is not affected by the choice of u_0, u_1, so this can be omitted from the cost function. Doing this, and collecting terms together suitably gives the cost function as:

$$\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}^T \left(\begin{bmatrix} B & 0 \\ AB & B \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} B & 0 \\ AB & B \end{bmatrix} + \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \right) \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} +$$

$$2x_0^T \begin{bmatrix} A \\ A^2 \end{bmatrix}^T \begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} B & 0 \\ AB & B \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \quad (39)$$

which has the form of (34), with $\theta = [u_0^T, u_1^T]^T$.
The constraints $|u_k| \leq U$ can be written as

$$u_k \leq U \quad \text{and} \quad u_k \geq -U \qquad (k = 0, 1) \quad (40)$$

which can be put into the form of (35):

$$\begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} \leq \begin{bmatrix} U \\ U \\ U \\ U \end{bmatrix} \quad (41)$$

Thus the predictive control problem has been rewritten in the standard form of a QP optimisation problem.

(c) A short horizon has the following disadvantages:

i. Excessively aggressive control actions; tendency to invert the plant model.
ii. More likely to be destabilising than a long horizon.
iii. 'Short-termism' likely to lead to dead-ends, ie it can drive the system into states from which no feasible solution exists.

Quite a few solutions were disguised versions of "A horizon of length 2 is too short because it is not long enough."

The horizon length N is limited by:

i. Computational complexity — the number of decision variables increases when N increases.
ii. The likelihood of infeasibility when a 'robust MPC' problem is posed, eg when unmeasured disturbances from a bounded set are allowed.
iii. The deterioration of prediction quality far into the future.

(The discussion could distinguish between the 'control horizon' which is relevant for point (i) and the 'prediction horizon' which is relevant for points (ii) and (iii).)

(d) Offset-free tracking of piecewise-constant set-points is obtained by one of the alternative strategies:

i. Penalise deviations of the controls from an 'ideal' value, so $u_k - u_0$ terms appear in the cost instead of u_k terms, and use an observer to estimate the value u_0 which corresponds to the current set-point.

ii. Penalise *changes* in the controls, $\Delta u_k = u_k - u_{k-1}$, rather than the controls themselves, and model tracking errors as piecewise-constant output disturbances.

(NB: Only one of these is expected from candidates. The second can be shown to be a special case of the first.)
