4M12 2008

1 (a) (i) $(\underline{a}\times(\underline{b}\times(\underline{a}\times\underline{c})))|_i = \varepsilon_{ijk} a_j \varepsilon_{klm} b_l \varepsilon_{mpq} a_p c_q$

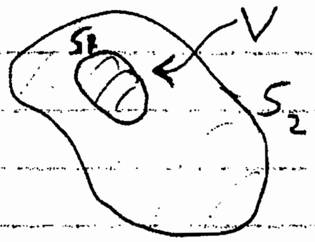
$$= \varepsilon_{ijk} a_j (\delta_{kp}\delta_{lq} - \delta_{kq}\delta_{lp}) b_l a_p c_q$$
$$= \varepsilon_{ijk} a_j a_k b_q c_q - \varepsilon_{ijk} a_j b_p a_p c_k$$
$$= 0 - (\underline{a}\cdot\underline{b})(\underline{a}\times\underline{c})|_i$$

(ii) $(\underline{a}\times\underline{b})\times(\underline{a}\times\underline{c})|_i = \varepsilon_{ijk}\varepsilon_{jlm} a_l b_m \varepsilon_{kpq} a_p c_q$

$$= \varepsilon_{jlm} a_l b_m a_p c_q (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})$$
$$= \varepsilon_{jlm} a_l b_m a_i c_j - \varepsilon_{jlm} a_l b_m a_j c_i$$
$$= (\underline{c}\cdot\underline{a}\times\underline{b})\, a_i - 0$$

Expression (i) is $\perp \underline{a}$, but (ii) is $/\!/ \underline{a}$, so they cannot be equal unless both expressions are zero. This requires $\underline{a}\perp\underline{b}$ or $\underline{a}/\!/\underline{c}$, and $\underline{a},\underline{b},\underline{c}$ all in one plane, unless $\underline{a}=0$ in which case they are trivially zero.

(b) The boundary of the volume V consists of both surfaces S_1, S_2. But care is needed with signs. The divergence theorem contains a surface integral over the total surface involving the outward-pointing normal vector. The outward-pointing normal over S_2 is in the usual direction for an integral over that surface, but on S_1 the outward-pointing normal relative to V is what would usually be called the inward-pointing normal to S_1. So the divergence theorem in its basic form states



$$\iiint_V \nabla\cdot\underline{g}\, dV = \iint_{S_2} \underline{g}\cdot d\underline{A} - \iint_{S_1} \underline{g}\cdot d\underline{A}$$

Apply this to the function $\underline{g} = \nabla\times\underline{f}$:

But $\nabla\cdot(\nabla\times\underline{f}) = 0$ for any $\underline{f}$ (data book, or use suffix notation to prove it as in the lecture notes).

So $$0 = \iiint_V \nabla\cdot(\nabla\times\underline{f})\,dV = \iint_{S_2} \nabla\times\underline{f}\cdot d\underline{A} - \iint_{S_1} \nabla\times\underline{f}\cdot d\underline{A}$$

So the two surface integrals are equal, as required.

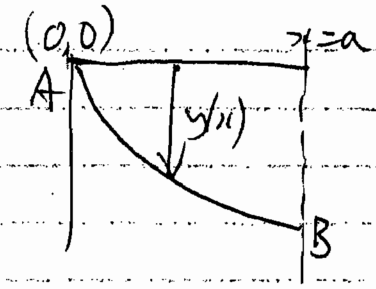
2. (a) $I = \int_0^a F(y, y', x)\, dx$

Let $y(x) \to y(x) + \delta y(x)$. Then to leading order

$$\delta I = \int_0^a \left\{ \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' \right\} dx$$

$$= \int_0^a \left\{ \frac{\partial F}{\partial x} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) \right\} \delta y \, dx + \left[\frac{\partial F}{\partial y'} \delta y \right]_0^a$$

Since δy is an arbitrary function, $\frac{\partial F}{\partial x} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$ for all x, a differential equation for $y(x)$.
Also the boundary term must vanish. If δy is not constrained at $x = a$, then $\frac{\partial F}{\partial y'} = 0$ there. This is the required free boundary condition.



(b)(i) By energy conservation, the speed v of the particle must satisfy

$$\tfrac{1}{2} m v^2 = m g y$$

$$\therefore v = \sqrt{2gy}$$

Total time of travel $T = \int_A^B \frac{ds}{v} = \int_0^a \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\, dx$

Since $\sqrt{2g}$ is a positive constant it can be ignored.
For minimum time, must minimise $\int_0^a \sqrt{\frac{1+y'^2}{y}}\, dx$

Apply result from (a) to get boundary condition at $x = a$: $\frac{\partial F}{\partial y'} = 0$ with $F = \sqrt{\frac{1+y'^2}{y}}$

$\therefore \frac{1}{\sqrt{y}}\frac{1}{2}(1+y'^2)^{-1/2}.2y' = 0$, ie require $y'=0$ at $x=a$

(ii) First integral from data sheet – $F(y,y') = \left(\frac{1+y'^2}{y}\right)^{1/2}$

does not involve x, so $F - y'\frac{\partial F}{\partial y'}$ = constant

$$\therefore \left(\frac{1+y'^2}{y}\right)^{1/2} - \frac{1}{y^{1/2}}y'^2(1+y'^2)^{-1/2} = E \text{ say}$$

$$\therefore 1+y'^2 - y'^2 = Ey^{1/2}(1+y'^2)^{1/2}$$

$$\therefore E^2y(1+y'^2) = 1$$

$$\therefore y' = \sqrt{\frac{1}{E^2y}-1} = \sqrt{\frac{1/E^2-y}{y}} \quad (1)$$

$$\therefore x - x_0 = \int\sqrt{\frac{y}{1/E^2-y}}\,dy \quad , x_0 \text{ a constant}$$

Set $y = \left(\frac{1}{E^2}\right)\sin^2\theta$, so $dy = \frac{1}{E^2}2\sin\theta\cos\theta\,d\theta$

Then RHS $= \int\frac{1/E\sin\theta \cdot 2/E^2\sin\theta\cos\theta\,d\theta}{1/E\cos\theta}$

$$= \frac{2}{E^2}\int\sin^2\theta\,d\theta = \frac{1}{E^2}\int(1-\cos 2\theta)d\theta$$

$$= \frac{1}{E^2}\left(\theta - \frac{1}{2}\sin 2\theta\right) = \frac{1}{2E^2}(2\theta - \sin 2\theta)$$

ie $x = b(2\theta - \sin 2\theta) + x_0$ with $B = 1/2E^2$

where $y = \frac{1}{E^2}\sin^2\theta = b(1-\cos 2\theta)$ as given

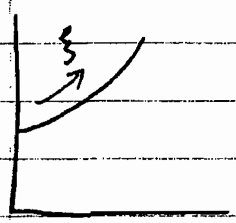
At $x=0$, $y=0$ so need $x_0 = 0$

At $x=a$, $y'=0$, so from (1) $y = 1/E^2 = 2b$,

ie $2\theta = \pi$. Thus $a = b(\pi - 0)$, so $b = a/\pi$

So $y(a) = 2b = \frac{2a}{\pi}$

3(a)



A variable ξ is introduced which increases along a line where

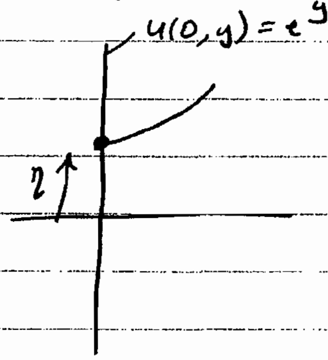
$$a = \frac{\partial x}{\partial \xi} \qquad b = \frac{\partial y}{\partial \xi} \qquad (1)$$

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c \quad \text{becomes} \quad \frac{\partial x}{\partial \xi}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial u}{\partial y} = c$$

i.e. $\dfrac{\partial u}{\partial \xi} = c$

This ordinary differential equation can then be integrated along the line given by equations (1) i.e. a "characteristic line".

(b) $\dfrac{\partial u}{\partial x} + xe^{-y}\dfrac{\partial u}{\partial y} = u \qquad x \geq 0$ for all y



$$u(0,y) = e^y$$

Using ch'ics

$$\frac{\partial x}{\partial \xi} = a = 1 \Rightarrow \frac{\partial y}{\partial \xi} = b = xe^{-y}$$

$\therefore \boxed{x = \xi}$ provided ξ chosen to be zero at $x=0$

Then $\dfrac{\partial y}{\partial \xi} = \xi e^{-y} \Rightarrow e^{y}\dfrac{\partial y}{\partial \xi} = \xi$ or $e^{y} = \dfrac{\xi^2}{2} + f(\eta)$

where η is a variable chosen to label the ch'ic

Conveniently, take $\eta = y$ at $x = \xi = 0$

$\therefore f(\eta) = e^{\eta}$ i.e. ch'ic is $\boxed{y = \ln\left(\frac{\xi^2}{2} + e^{\eta}\right)}$

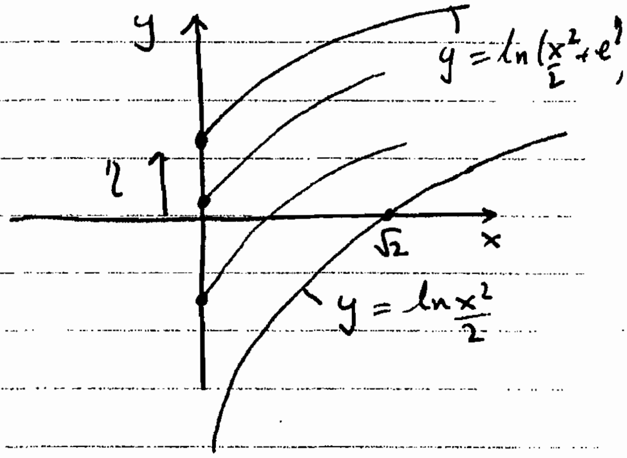
Thus $\xi = x$, $\eta = \ln\left(e^{y} - \frac{x^2}{2}\right)$

Then $\dfrac{\partial u}{\partial \xi} = u \Rightarrow \ln u = \xi + g(\eta)$ or $u = A(\eta)e^{\xi}$

B.C. $u(x=0) = e^{y} \Rightarrow e^{\eta} = A(\eta) \therefore \underline{u = e^{\eta+\xi} = e^{x}\left(e^{y} - \frac{x^2}{2}\right)}$

(c) Ch'ics are $\eta = \text{const}$ where $e^y - \frac{x^2}{2} = e^\eta \Rightarrow y = \ln\left(\frac{x^2}{2} + e^\eta\right)$

Solution is valid only in that part of the domain which can be reached by a ch'ic emanating from the y-axis.



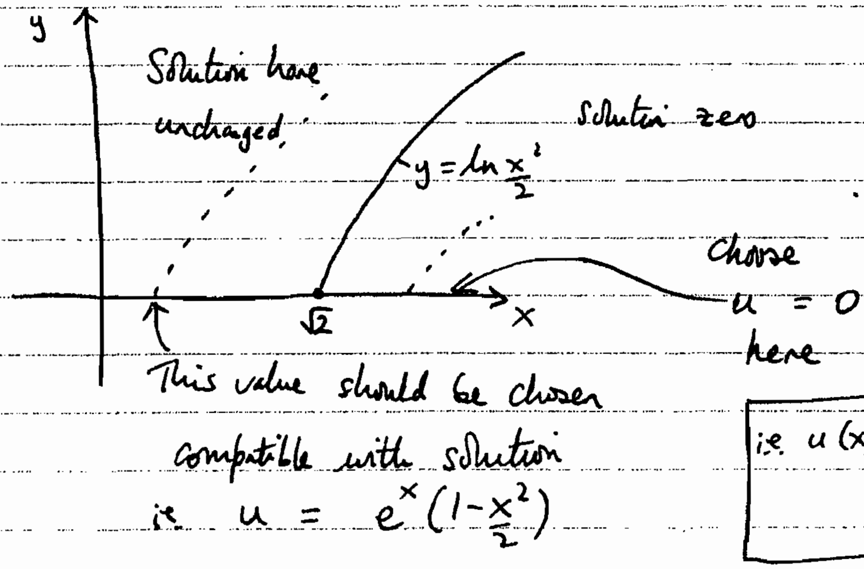
As $\eta \to -\infty$ the limiting ch'ic is

$$y = \ln\left(\frac{x^2}{2} + 0\right)$$

The solution is thus appropriate for

$$y > \ln\frac{x^2}{2}$$

(d)



Solution is always

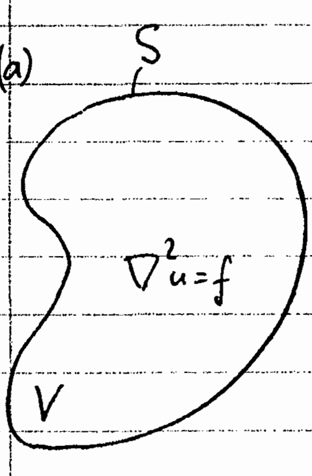
$$u = A(\eta)e^{\xi}$$

$\therefore$ In $u = 0$ region need $A = 0$

$\Rightarrow u(x,0) = 0$

$$\boxed{\begin{aligned} \text{i.e. } u(x,0) &= e^x\left(1 - \frac{x^2}{2}\right) & x < \sqrt{2} \\ &= 0 & x > \sqrt{2} \end{aligned}}$$

4(a)



$G(\underline{x}, \underline{x}_0)$ satisfies $\nabla^2 G = \delta(\underline{x} - \underline{x}_0)$ in V

(∇ = differentiation w.r.t. $\underline{x}$)

Now

$$\int_V (u\nabla^2 G - G\nabla^2 u)\,dV = \int_V \underline{\nabla}\cdot(u\nabla G - G\nabla u)\,dV$$

$$= \int_S \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right)dS \quad \text{by divergence thm.}$$

$$\Rightarrow \int \left[u(\underline{x})\,\delta(\underline{x} - \underline{x}_0) - G(\underline{x}, \underline{x}_0) f(\underline{x})\right]dV = \int_S \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right)dS$$

i.e.
$$u(\underline{x}_0) = \int_V G(\underline{x}, \underline{x}_0) f(\underline{x})\,dV + \int_S \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right)dS$$

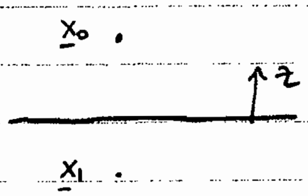
The boundary condition for G is chosen to be
$G(\underline{x}, \underline{x}_0) = 0$ for $\underline{x} \in S$ and for each $\underline{x}_0$ in V

Then the representation theorem becomes

$$u(\underline{x}_0) = \int_V G(\underline{x}, \underline{x}_0) f(\underline{x})\,dV + \int_S U(\underline{x})\frac{\partial G(\underline{x}, \underline{x}_0)}{\partial n}\,dS$$

which is the solution for u.

(b)



Seek $\nabla^2 G = \delta(\underline{x} - \underline{x}_0)$ for $z > 0$

and $G(\underline{x}, \underline{x}_0) = 0$ when $z = 0$

Datacard $\Rightarrow$ one solution of $\nabla^2 u = \delta(\underline{x} - \underline{x}_0)$ is

$$-\frac{1}{4\pi|\underline{x} - \underline{x}_0|}$$

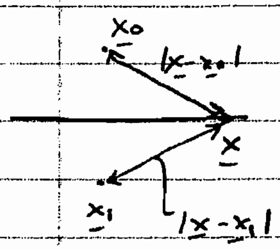
$$\therefore \nabla^2\left\{-\frac{1}{4\pi|\underline{x} - \underline{x}_0|} + \frac{1}{4\pi|\underline{x} - \underline{x}_1|}\right\} = \delta(\underline{x} - \underline{x}_0) - \delta(\underline{x} - \underline{x}_1)$$

$\to 0$ if $\underline{x}_1$ outside $z_1 > 0$

If we choose $\underline{x}_1 = (x_0, y_0, -z_0)$ then it is outside $z_0 > 0$

and on $z=0$

$$G(\underline{x},\underline{x}_0) = -\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(-z_0)^2}} + \frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+z_0^2}}$$

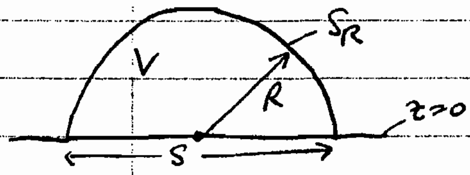


$$= 0$$

$\Rightarrow$ G is required Gn Fn.

(c) Applying the rep^n theorem to the volume shown

$$u(\underline{x}_0) = \int_S u\frac{\partial G}{\partial n}\,dS + \int_{S_R}\left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right)dS$$



Assuming u decays sufficiently quickly that

$$\int_{S_R}(\quad)\,dS \to 0 \text{ as } R\to\infty \qquad (A)$$

then $$u(\underline{x}_0) = \int_{\text{all } x,y} U(x,y)\,-\frac{\partial G}{\partial z}\bigg|_{z=0} dx\,dy$$

$$G = -\frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}} + \frac{1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}$$

$$\Rightarrow \frac{\partial G}{\partial z} = \frac{1}{2.4\pi.}\frac{2(z-z_0)}{\left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{3/2}} - \frac{1}{2.4\pi}\frac{2(z+z_0)}{\left[(x+x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{3/2}}$$

$$\Rightarrow -\frac{\partial G}{\partial z}\bigg|_{z=0} = \frac{2z_0}{4\pi\left[(x-x_0)^2+(y-y_0)^2+z_0^2\right]^{3/2}}$$ As required

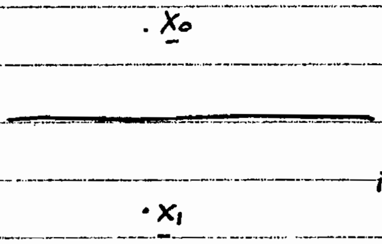
Provided $\iint U\,dx\,dy$ bounded then $u = O\left(\frac{z_0}{R^3}\right)$ at large R

$$\Rightarrow \int_{S_R}\left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right)dS = O\left(\frac{1}{R^2}\cdot\frac{1}{R^2}\cdot R^2\right)$$

$\to 0$ as $R\to\infty$ (consistent with A)

This is sufficient but not necessary.

(d)



For V, $\dfrac{\partial G'}{\partial n} = 0$ on $z = 0$

i.e. $G' = -\dfrac{1}{4\pi|\underline{x}-\underline{x}_0|} - \dfrac{1}{4\pi|\underline{x}-\underline{x}_1|}$

and $V(\underline{x}_0) = \displaystyle\int_{\text{all } x,y} U G' \, dx\,dy \simeq -\frac{1}{2\pi|\underline{x}_0|} \int U \, dx\,dy$

& $u(\underline{x}_0) \sim \dfrac{z_0}{2\pi|\underline{x}_0|^3} \displaystyle\int U \, dx\,dy \quad \Rightarrow \quad V \gg u$

[Aliter G has two pt sources of opposite sign $\Rightarrow$ looks like dipole
G' .. .. same sign $\Rightarrow$ source]

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