4M13 — COMPLEX ANALYSIS AND OPTIMISATION p1 IIB 4M13
Solutions DR GT PARKS 2008
FINAL VERSION

4M13 Exam solutions 2008 Dr. Gábor Csányi

1.

a) i. If f has a zero of order k at z_0, then it can be written as $f = (z - z_0)^k g(z)$ with $g(z) \neq 0$ and analytic near z_0. Therefore $f'(z) = k(z - z_0)^{k-1}g(z) + (z - z_0)^k g'(z)$, hence

$$\frac{f'(z)}{f(z)} = \frac{k}{z - z_0} + \frac{g'(z)}{g(z)}$$ (30%)

which has a simple pole at z_0 with residue k because g'/g is analytic near z_0.

ii. In this case, f can be written as $f(z) = g(z)/(z - z_0)^m$, where again $g(z) \neq 0$ and analytic near z_0. Hence

$$\begin{aligned} f'(z) &= \frac{-mg(z)}{(z - z_0)^{m+1}} + \frac{g'(z)}{(z - z_0)^m} \\ \frac{f'(z)}{f(z)} &= \frac{-m}{z - z_0} + \frac{g'(z)}{g(z)} \end{aligned}$$ (30%)

which has a simple pole at z_0 with residue $-m$. [60%]

b) The residue theorem says that the integral of a function along a closed contour on which there are no poles is equal to $2\pi i \times$ (sum of residues of poles inside the contour). According to i. and ii. above, the sum of the residues of the poles of f'/f inside S are $\sum_i z_i k_i - \sum_j w_j m_j$ where the i sum runs over the zeros of f at $z = z_i$ and the j sum runs over the poles of f at $z = w_j$. Hence

$$\begin{aligned} \frac{1}{2\pi i} \int_{dS} \frac{f'(z)}{f(z)} dz &= \sum_i z_i k_i - \sum_j w_j m_j \\ &= N - P \end{aligned}$$ [25%]

c) The polynomial has no poles and two zeros at $z = \pm 1/2$, so using iii), we just need to integrate f'/f on the unit circle. The derivative is $f'(z) = 8z$. The two poles of f'/f are at $z = \pm 1/2$. The integral in b) is

$$\oint \frac{8z}{4z^2 - 1} dz$$

The residues at the poles are: $\lim_{z \to 1/2} \frac{8z(z - 1/2)}{4(z - 1/2)(z + 1/2)} = 1$ and similarly for the other pole, thus the sum of the residues is $2 \times 2\pi i$, corresponding to the 2 zeros of f. [15%]

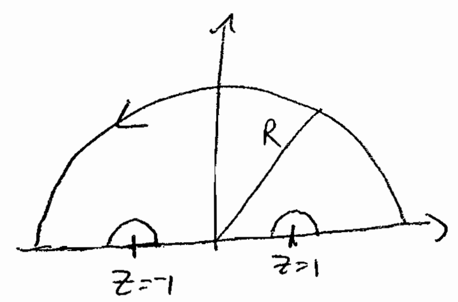
2.

a) Use the substitution $z = x$ and will integrate on a contour that consists of the real axis and a very large semicircle. The complex integral on the real axis is

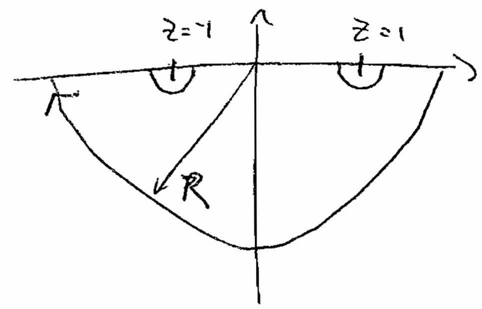
$$\begin{aligned} I &= \frac{1}{2i} \int \frac{z(e^{i\pi z} - e^{-i\pi z})}{1 - z^2} dz \\ &= \frac{1}{2i} \int \frac{z e^{i\pi z}}{1 - z^2} dz - \frac{1}{2i} \int \frac{z e^{-i\pi z}}{1 - z^2} dz \end{aligned}$$

We want to use Jordan's Lemma, so complete contours for the two integrals slightly differently: for the first integral, we complete the contour in the upper half plane, for the second in the lower half plane.

1st integral



2nd integral



Jordan's Lemma states that if a is real and $a>0$

$$\int_\Gamma e^{iaz} g(z)\,dz \;\rightarrow\; 0 \qquad \text{as } R\rightarrow\infty\ldots$$

where Γ is a semicircle of radius R in the upper half plane and $g(z)\rightarrow 0$ as $z\rightarrow\infty$. If $a<0$, then the integral vanishes on a very large semicircle in the lower half plane.

Hence the integrals vanish on both semicircles, and the real integral is equal to the $2\pi i\times$ (sum of the residues). Note that the poles of the integrand are at $z=\pm 1$, i.e. on the contour, so only half of the residue contributes. Not also that the two complex integrals are equal. The residues are

$$\lim_{z\rightarrow\pm 1}\frac{(z\pm 1)ze^{i\pi z}}{(1+z)(1-z)} \;=\; \frac{1}{2}$$

so the real integral is $\frac{2\pi i}{2i}\left(\frac{1}{2}+\frac{1}{2}\right)=\pi$ [50%]

b) Convert the integral to a complex one, with $z=e^{ix}$ (with $dz=izdx$) and integrate on the unit semicircle in the upper half plane. Since the original integral is an even function of x, we can extend the complex integral to the full unit circle and divide the answer by 2. So

$$\begin{aligned}\int_0^\pi \frac{dx}{1+b\cos^2 x} &= \frac{1}{2i}\oint \frac{dz}{z\left[1+\frac{b}{4}\left(z+\frac{1}{z}\right)^2\right]}\\ &= \frac{1}{2i}\oint \frac{zdz}{\frac{b}{4}z^4+(\frac{b}{2}+1)z^2+\frac{b}{4}}\end{aligned}$$

Using the quadratic formula in z^2 for the denominator, it has four roots, satisfying $z^2=-1+\frac{2}{b}(\pm\sqrt{b+1}-1)$, with two of them (corresponding to the $+$ sign) inside the unit circle. Let us label these roots with $z=\pm i\gamma_+$ with the other roots at $z=\pm i\gamma_-$. The residues at the poles inside the unit circle are

$$\lim_{z\rightarrow\pm i\gamma_+}\frac{(z\pm i\gamma_+)z}{\frac{b}{4}(z^2+\gamma_+^2)(z^2+\gamma_-^2)} \;=\; \frac{2/b}{\gamma_-^2-\gamma_+^2}$$

Now $\gamma_-^2-\gamma_+^2=\frac{4}{b}\sqrt{b+1}$, so $2\pi i\times$ the sum of the two residues is $\frac{\pi}{\sqrt{b+1}}$. [50%.

Q3 (a) If the design is 'fully stressed' then the cable tensile stress $\sigma = \sigma_d$ (the design strength). Therefore the cable cross-sectional area

$$A = \frac{T}{\sigma_d}$$

The volume of steel in each cable span is

$$V = As = \frac{T}{\sigma_d}s = \frac{wL^2}{8d\sigma_d}L\left(1+\frac{8d^2}{3L^2}\right)$$

hence the cost of cable in each span is $= \dfrac{wL^3}{8d\sigma_d}\left(1+\dfrac{8d^2}{3L^2}\right)c_{cable}$

If there are n spans then $n+1$ towers are required. Treating n as a continuous variable the total cost of the bridge is:

$$f = n\frac{wL^3}{8d\sigma_d}\left(1+\frac{8d^2}{3L^2}\right)c_{cable} + (n+1)C_{tower}$$

$$= \frac{b}{L}\frac{wL^3}{8d\sigma_d}\left(1+\frac{8d^2}{3L^2}\right)c_{cable} + \left(\frac{b}{L}+1\right)C_{tower}$$

$$= \frac{bwc_{cable}}{8\sigma_d}\left(\frac{L^2}{d}+\frac{8}{3}d\right) + C_{tower}\left(\frac{b}{L}+1\right)$$ [15%]

(b) Substituting the values given

$$f = \frac{10{,}000\times160{,}000\times100{,}000}{8\times1000\times10^6}\left(\frac{L^2}{d}+\frac{8}{3}d\right)+6.532\times10^6\left(\frac{10{,}000}{L}+1\right)$$

$$= 20{,}000\left(\frac{L^2}{d}+\frac{8}{3}d\right)+6.532\times10^6\left(\frac{10{,}000}{L}+1\right)$$

$$= 10^4\left(\frac{2L^2}{d}+\frac{16d}{3}+\frac{6.532\times10^6}{L}+653.2\right)$$ (5%)

Thus

$$\nabla f = \left(\frac{\partial f}{\partial L} \quad \frac{\partial f}{\partial d}\right)^T = 10^4\left[\frac{4L}{d}-\frac{6.532\times10^6}{L^2} \quad -\frac{2L^2}{d^2}+\frac{16}{3}\right]^T$$ (10%)

$$\mathbf{H} = \begin{bmatrix} \dfrac{\partial^2 f}{\partial L^2} & \dfrac{\partial^2 f}{\partial L\partial d} \\ \dfrac{\partial^2 f}{\partial d\partial L} & \dfrac{\partial^2 f}{\partial d^2} \end{bmatrix} = 10^4\begin{bmatrix} \dfrac{4}{d}+\dfrac{2\times6.532\times10^6}{L^3} & -\dfrac{4L}{d^2} \\ -\dfrac{4L}{d^2} & \dfrac{4L^2}{d^3} \end{bmatrix}$$ (10%)

Starting at $L = 2000$ m and $d = 100$ m:

$$\mathbf{x_0} = (2000,100)^T \quad f(\mathbf{x_0}) = £\ 844.5 \text{ million}$$

$$\nabla f(\mathbf{x_0}) = 10^6(0.7837 \quad -7.947)^T \text{ and } \mathbf{H}(\mathbf{x_0}) = \begin{bmatrix} 416.33 & -8000 \\ -8000 & 160000 \end{bmatrix}$$

$$\mathbf{d_0} = -\nabla f(\mathbf{x_0}) = 10^6(-0.7837 \quad 7.947)^T$$

(10%)

$$\alpha_0 = \frac{-\mathbf{d}_0{}^T \nabla f(\mathbf{x_0})}{\mathbf{d}_0{}^T \mathbf{H}\, \mathbf{d}_0}$$

$$= -\frac{0.7837^2 + 7.947^2}{(-0.7837 \quad 7.947)\begin{bmatrix} 416.33 & -8000 \\ -8000 & 160000 \end{bmatrix}\begin{pmatrix} -0.7837 \\ 7.947 \end{pmatrix}}$$

$$= \frac{63.769}{-63902 \times -0.7837 + 1277790 \times 7.947} = 6.249 \times 10^{-6}$$

(10%)

$$\mathbf{x_1} = \mathbf{x_0} + \alpha_0 \mathbf{d_0} = (2000 - 4.9,\ 100 + 49.7)^T = (1995.1,\ 149.7)^T$$

(5%)

Check: $f(\mathbf{x}_1) = £\ 579.0$ million and this is an improved design **[50%]**

(c) For minimum $\nabla f = 0$

So from expression for ∇f

$$4L^3 = 6.532 \times 10^6 d \text{ and } 6L^2 = 16d^2$$

Thus $4L^3 = 6.532 \times 10^6 \times \sqrt{\frac{3}{8}}L$

Hence $L = 1000\,\text{m}$

and $d = \sqrt{\frac{3}{8}}L = 612.37\,\text{m}$

For these values $\mathbf{H} = \begin{bmatrix} 196.0 & -106.7 \\ -106.7 & 174.2 \end{bmatrix}$ which is clearly positive definite (top left corner entry is > 0 and so is its determinant [22758]). Thus this solution is a minimum.

Check: $f = £\ 137.2$ million for these values of L and d. **[20%]**

(d) The convergence rate of the steepest descent method will be poor on this objective function which is far from quadratic in form. Thus it is not surprising that in one iteration it makes only limited progress towards the optimum.

The cost model is not realistic because in reality the cost of the towers will increase significantly with increasing height d. Towers 600 m higher than a deck of 1000 m span do not make any sense. The Humber Bridge has a span of ~1400 m and its towers are only ~130 m higher than the deck.

[15%]

4 (a) The weight (mass) is proportional to the material volume so it is sufficient to minimize V. Thus the problem is

$$\text{Minimize } f = V = \pi L(R_o^2 - R_i^2)$$
$$\text{subject to } g_1 = P - \sigma_y \pi (R_o^2 - R_i^2) \leq 0$$
$$g_2 = P - \frac{\pi^3 E}{16L^2}(R_o^4 - R_i^4) \leq 0$$
$$g_3 = R_i - R_o \leq 0 \quad \text{ie } R_o \geq R_i$$
$$g_4 = -R_i \leq 0 \quad \text{ie } R_i \geq 0$$

[10%]

(b) Consider g_1 first:

$$R_o^2 - R_i^2 \geq \frac{P}{\pi \sigma_y} = 7{\cdot}5\times10^{-3} \text{ for values given}$$

$$\therefore R_o \geq \left[R_i^2 + 7{\cdot}5\times10^{-3}\right]^{1/2} \quad (1) \qquad (5\%)$$

Consider g_2:

$$R_o^4 - R_i^4 \geq \frac{16L^2P}{\pi^3 E} = \frac{16\pi^2 \times 7{\cdot}5\times10^{-3}\pi\sigma_y}{\pi^3 \times 1280\sigma_y}$$
$$= 9{\cdot}375\times10^{-5}$$

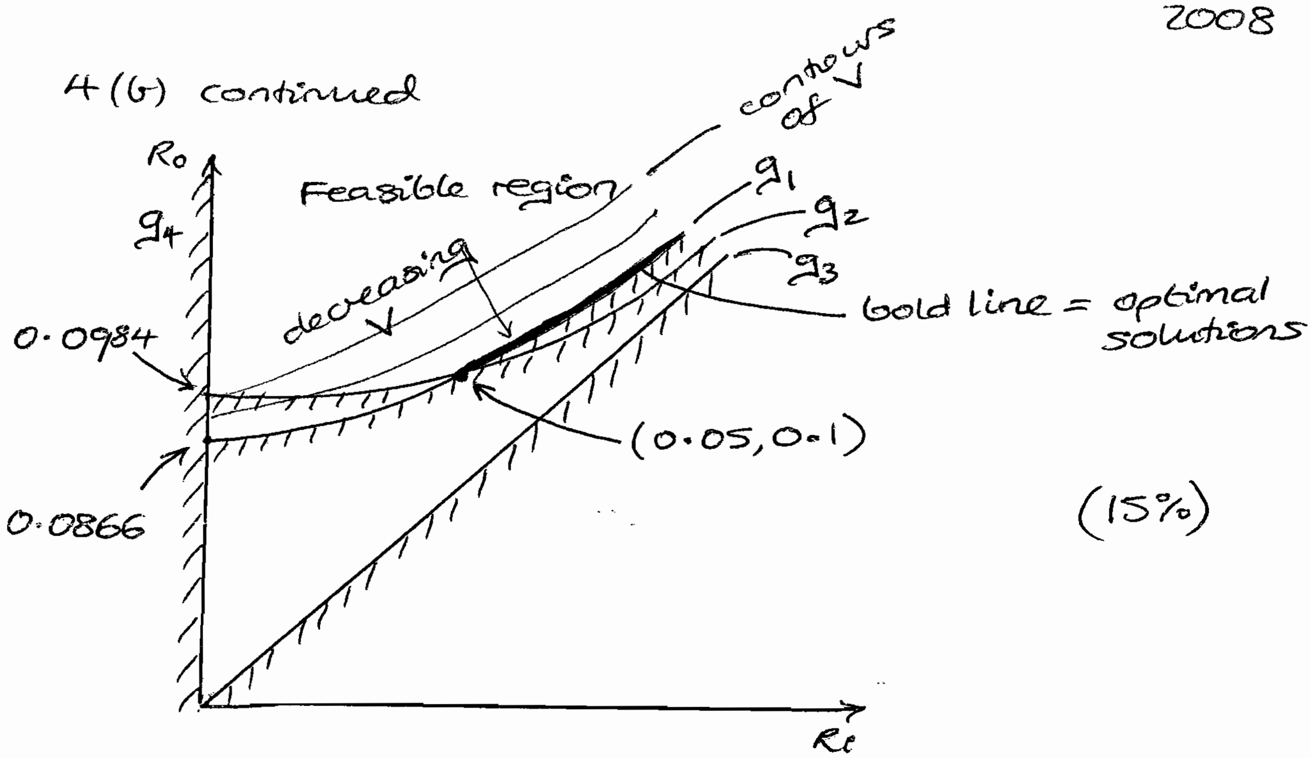
$$\therefore R_o \geq \left[R_i^4 + 9{\cdot}375\times10^{-5}\right]^{1/4} \quad (2) \qquad (5\%)$$

(1) and (2) cross when

$$R_i^4 + 9{\cdot}375\times10^{-5} = \left[R_i^2 + 7{\cdot}5\times10^{-3}\right]^2$$
$$= R_i^4 + 0{\cdot}015R_i^2 + 5{\cdot}625\times10^{-5}$$
$$\therefore 0{\cdot}015R_i^2 = 3{\cdot}75\times10^{-5}$$
$$\therefore R_i = 0{\cdot}05 \text{ m}$$
$$\therefore R_o = \left[0{\cdot}05^2 + 7{\cdot}5\times10^{-3}\right]^{1/2} = 0{\cdot}1 \text{ m} \qquad (10\%)$$

Hence (see next page)

4 (b) continued



(15%)

By inspection contours of V are parallel to g_1. (5%)

Looking at the figure above, there are multiple optima. Any solution for which $R_i \geq 0.05$ m and $R_o = \left[R_i^2 + 7.5\times10^{-3}\right]^{1/2}$ gives the same minimum volume. (5%)

By inspection g_1 is active at the optimum, and g_2 is active at the solution $[0.05, 0.1]$. (5%)
g_3 and g_4 are never active.

[50%]

(c) $$L = \pi L(R_o^2 - R_i^2) + \mu_1\left[P - \sigma_y\pi(R_o^2 - R_i^2)\right] + \mu_2\left[P - \frac{\pi^3 E}{16L^2}(R_o^4 - R_i^4)\right]$$

$$\frac{\partial L}{\partial R_o} = 2\pi L R_o - 2\mu_1\sigma_y\pi R_o - \frac{1}{4}\mu_2\frac{\pi^3 E}{L^2}R_o^3 = 0 \quad (1)$$

$$\frac{\partial L}{\partial R_i} = -2\pi L R_i + 2\mu_1\sigma_y\pi R_i + \frac{1}{4}\mu_2\frac{\pi^3 E}{L^2}R_i^3 = 0 \quad (2)$$

$$\mu_1\left(P - \sigma_y\pi(R_o^2 - R_i^2)\right) = 0 \quad (3)$$

$$\mu_2\left(P - \frac{\pi^3 E}{16L^2}(R_o^4 - R_i^4)\right) = 0 \quad (4)$$

(10%)

4(c) continued

Comment: don't need to include g_3 and g_4 as we know they must be inactive at optimum.

Case (i) $\mu_1 = 0, \mu_2 = 0$

(1) $\Rightarrow \quad 2\pi L R_o = 0 \quad \Rightarrow R_o = 0$

(2) $\Rightarrow \quad -2\pi L R_i = 0 \quad \Rightarrow R_i = 0$ (5%)

if $R_o = R_i = 0$ g_1 and g_2 are violated

$\therefore$ impossible

Case (ii) $\mu_1 = 0, \mu_2 > 0$

(1) $\Rightarrow \quad 2\pi L R_o - \frac{1}{4}\mu_2 \frac{\pi^3 E}{L^2} R_o^3 = 0$

$\Rightarrow \quad R_o = 0 \quad$ or $\quad R_o^2 = \dfrac{8L^3}{\mu_2 \pi^2 E}$

(2) $\Rightarrow \quad 2\pi L R_i - \frac{1}{4}\mu_2 \frac{\pi^3 E}{L^2} R_i^3 = 0$ (5%)

$\Rightarrow \quad R_i = 0 \quad$ or $\quad R_i^2 = \dfrac{8L^3}{\mu_2 \pi^2 E}$

$\therefore$ $R_i = R_o$ which means g_1 and g_2 are violated $\therefore$ impossible

Case (iii) $\mu_1 > 0, \mu_2 = 0$

(1) $\Rightarrow \quad 2\pi L R_o - 2\mu_1 \sigma_y \pi R_o = 0$

$\Rightarrow \quad R_o = 0 \quad$ or $\quad \mu_1 = L/\sigma_y$

(2) $\Rightarrow \quad 2\pi L R_i - 2\mu_1 \sigma_y \pi R_i = 0$

$\Rightarrow \quad R_i = 0 \quad$ or $\quad \mu_1 = L/\sigma_y$

For $\mu_1 = L/\sigma_y$ (clearly +ve) case (10%)

(3) $\Rightarrow \quad P = \sigma_y \pi (R_o^2 - R_i^2)$

$\Rightarrow$ infinite solutions as long as R_o and R_i satisfy g_1 and do not violate g_2.

4(c) continued

Case (iv) $\mu_1 > 0$, $\mu_2 > 0$

(1) $\Rightarrow$ $R_o = 0$ or

$$2\pi(L - \mu_1 \sigma_y) - \frac{1}{4}\mu_2 \frac{\pi^3 E}{L^2} R_o^2 = 0 \qquad (A)$$

Similarly (2) $\Rightarrow$ $R_i = 0$ or

$$2\pi(L - \mu_1 \sigma_y) - \frac{1}{4}\mu_2 \frac{\pi^3 E}{L^2} R_i^2 = 0 \qquad (B)$$

Equating (A) and (B) and simplifying

$$\mu_2 R_o^2 = \mu_2 R_i^2$$ (10%)

$\Rightarrow$ $\mu_2 = 0$ i.e. no solution for $\mu_2 > 0$

or $R_o = R_i$ which means g_1 and g_2 are violated $\therefore$ impossible

Hence only solution occurs in case (iii)

[40%]