

ENGINEERING TRIPOS PART IIB

Tuesday 22 April 2008 9 to 10.30

Module 4A8

ENVIRONMENTAL FLUID MECHANICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments: 4A8 Data Card (5 pages)

STATIONERY REQUIREMENTS

Single-sided script paper

SPECIAL REQUIREMENTS

Engineering Data Book

CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) Provide a brief qualitative description of:

(i) geostrophic flow and the Ekman spiral; [20%]

(ii) Brunt-Väisälä frequency; [20%]

(iii) the turbulent closure problem; [20%]

(iv) sources and sinks of turbulent kinetic energy in atmospheric flows. [20%]

(b) Show that the Dry Adiabatic Lapse Rate is given by

$$\frac{dT}{dz} = -\frac{g}{c_p}$$

and discuss its significance. [20%]

2 For a boundary layer with stable density stratification the velocity gradient is given by

$$\frac{dU}{dz} = \frac{u_*}{\kappa z} \left(1 + 4.7 \frac{z}{L} \right)$$

where $\kappa \approx 0.4$ and L is the Monin-Obukhov length. The other variables have their usual meaning.

(a) Find U as a function of z if $U = 0$ at $z = z_0$, where z_0 is the surface roughness height. [20%]

(b) When $z_0 = 0.1$ m, measurements show that the velocity U is 2.0 m s^{-1} at $z = 5$ m and 3.0 m s^{-1} at $z = 10$ m. Determine L , the friction velocity, the surface shear stress, and the surface heat flux. The ambient temperature near the ground is 280 K. [40%]

(c) Estimate the rate of turbulent kinetic energy production due to mean shear and the rate of energy dissipation via buoyancy at $z = 5$ m. You may take ρu_*^2 as the scaling of the relevant Reynolds stress. [20%]

(d) Far away from the surface, the velocity gradient is given by

$$\frac{dU}{dz} = \frac{u_*}{\kappa z} \left(4.7 \frac{z}{L} \right)$$

Show that the ratio of energy dissipation via buoyancy to mechanical generation of turbulent kinetic energy is constant and find this constant. [20%]

(TURN OVER)

3 Consider a Gaussian plume from a source at $(x, y, z) = (0, 0, 0)$ emitting $Q \text{ kg s}^{-1}$ of pollutant spreading in two dimensions (y and z) in uniform wind of velocity U aligned with the x -direction. The dispersion coefficient σ is identical in both y and z directions and increases linearly with x . Ignore any effects from solid surfaces. The transport equation for the variance of the scalar fluctuations g is found in the Data Card. The mean centreline concentration at a downwind distance x is Φ_0 , K is the turbulent diffusivity, u' the characteristic large-scale turbulent velocity (assumed uniform and isotropic everywhere), and T_{turb} the eddy turn-over time.

(a) Discuss if the assumption of equal dispersion coefficients is in general valid for turbulent flows close to the surface under neutral stability conditions. [20%]

(b) Neglecting gradients in the x -direction and assuming that the dissipation of g at $(x, y, z) = (x, 0, 0)$ is equal to the production of g at $(x, y, z) = (x, \sigma, \sigma)$, show that

$$\frac{g^{1/2}}{\Phi_0} = \frac{(2KT_{turb})^{1/2}}{\sigma} e^{-1}.$$

[50%]

(c) If the ratio $g^{1/2} / \Phi_0$ is, from measurement, equal to 0.25, find an expression for K in terms of u' and σ including any numerical constants. [30%]

4 A typical urban street canyon between buildings of height H is shown in Fig. 1. Inert pollutant is being emitted at a rate of Q per unit area (units: $\text{kg s}^{-1} \text{m}^{-2}$) at the street surface over the length L and width W of the canyon. The air flow above the canyon is turbulent and normal to the canyon and carries b kg m^{-3} of the pollutant. It hence exchanges mass with the street canyon at a rate per unit of exchange area given by $\dot{m}' = u(c - b)$, where c is the average concentration of the pollutant in the canyon (units of $\dot{m}'$: $\text{kg s}^{-1} \text{m}^{-2}$) and u is a characteristic velocity for the exchange.

(a) Derive the following equation for the rate of change of c starting from first principles:

$$\frac{dc}{dt} = \frac{Q}{H} - \frac{u}{H}(c - b) .$$

[20%]

(b) Assume that c has reached its steady-state value. Q is then suddenly decreased to zero. For $H = 10$ m, $u = 0.5$ m s^{-1} , $b = 0$ and $Q = 10^{-8}$ $\text{kg s}^{-1} \text{m}^{-2}$, find the time taken for the street canyon to be purged to a pollutant concentration 5% of its value when the emission stopped.

[50%]

(c) List possible factors that may affect the velocity u .

[30%]

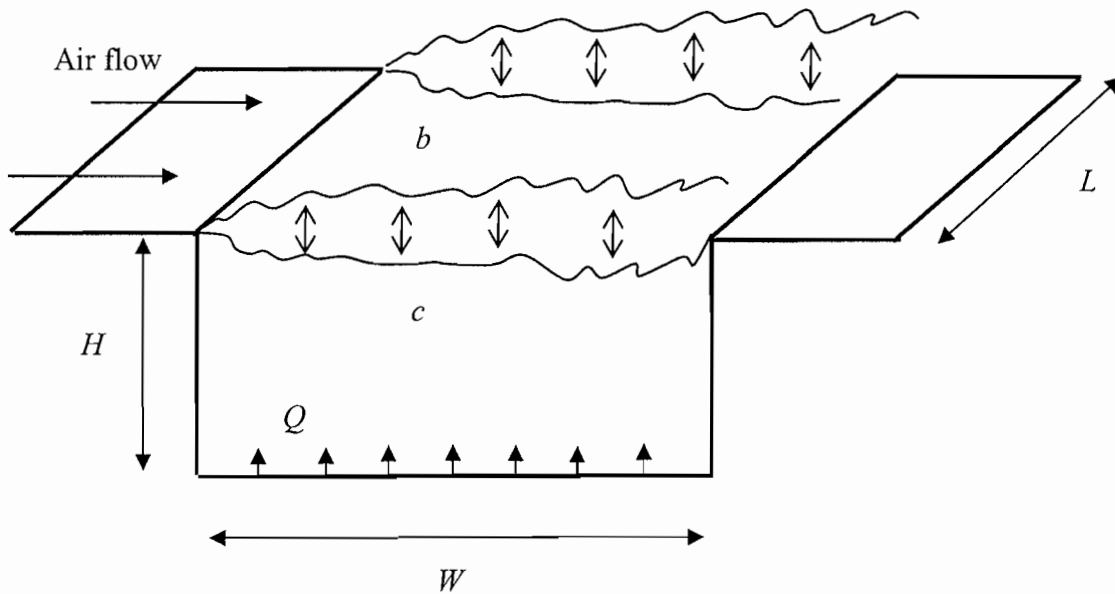


Fig. 1

END OF PAPER

4A8: Environmental Fluid Mechanics

Part I: Turbulence and Fluid Mechanics

DATA CARD

Rotating Flows

Geostrophic Flow

$$-\frac{1}{\rho} \nabla p = 2\Omega \times \underline{u}$$

Ekman Layer Flow

$$-2\Omega_z v = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2}$$

$$2\Omega_z u = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \frac{\partial^2 v}{\partial z^2}$$

$$0 = -\frac{1}{\rho} \frac{\partial p}{\partial z}$$

OR

$$-2\Omega_z v = \nu \frac{\partial^2 u}{\partial z^2}$$

$$-2\Omega_z (u_g - u) = \nu \frac{\partial^2 v}{\partial z^2}$$

GEOSTROPHIC VELOCITY

Solution is

$$u = u_g \left[1 - e^{-z/\Delta} \cos \frac{z}{\Delta} \right]$$

$$v = u_g e^{-z/\Delta} \sin \frac{z}{\Delta}$$

$$\Delta = \left(\frac{\nu}{\Omega_z} \right)^{1/2}$$

Turbulent Flows – Incompressible

Continuity Equation $\nabla \bullet \underline{U} = \frac{\partial U_i}{\partial x_i} = 0$

Momentum Equation $\rho \frac{DU}{Dt} = -\nabla P + \mu \nabla^2 \underline{U} + \underline{F}$

$$\rho \frac{DU_i}{Dt} = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 U_i}{\partial x_j^2} + F_i$$

Enthalpy Equation $\rho c_p \frac{DT}{Dt} = -k \frac{\partial^2 T}{\partial x_i^2}$

Reynolds Transformation $U_i = \overline{U_i} + u_i \text{ etc}$

Reynolds Stress $= -\overline{\rho u_i u_j}$

Reynolds Heat Flux $= -\overline{\rho c_p u_j \Theta}$

Turbulent Kinetic Energy Equation

$$\frac{D}{Dt} \frac{q^2}{2} = -\overline{u_i u_k} \frac{\partial \overline{U_i}}{\partial x_k} - \nu \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) \left(\frac{\partial u_i}{\partial x_k} \right) + \frac{\overline{f_i u_i}}{\rho} + \text{transport of kinetic energy forms}$$

In flows with thermally driven motion

$$\frac{\overline{f_i u_i}}{\rho} = \frac{g}{T} \bullet \overline{\Theta u_i} = \frac{gQ}{\rho c_p T}$$

$i = \text{Vertical direction}$
 $Q = \text{surface heat flux}$

Dissipation of turbulent kinetic energy $\varepsilon \approx \frac{u'^3}{\ell}$

Kolmogorov microscale $\eta = \left(\frac{\nu^3}{\varepsilon} \right)^{1/4}$

Taylor microscale (λ) $\varepsilon = 15\nu \frac{u'^2}{\lambda^2}$ $\nu = \text{kinematic viscosity}$

Density Influenced Flows

Atmospheric Boundary Layer

$$\left. \frac{dT}{dz} \right|_{\text{NEUTRAL STABILITY}} = -\frac{g}{C_p} = \left. \frac{dT}{dz} \right|_{\text{DALR}}$$

$$R_i = \frac{g}{T} \frac{\left. \frac{dT}{dz} - \left. \frac{dT}{dz} \right|_{\text{DALR}} \right|}{\left(\left. \frac{dU}{dz} \right|_{\text{DALR}} \right)^2} = \text{RICHARDSON NUMBER}$$

Neutral Stability

$$U = \frac{u_*}{\kappa} \ln \frac{z}{z_0}; \quad \frac{dU}{dz} = \frac{u_*}{\kappa z}$$

$$u_* = \sqrt{\frac{\tau_w}{\rho}}; \quad \kappa = \text{Von Karman Constant} = 0.40$$

Non-Neutral Stability

$$L = \text{Monin-Obukhov length} = -\frac{u_*^3}{\kappa \frac{g}{T} \frac{Q}{\rho c_p}}$$

Q = surface heat flux

$$\frac{dU}{dz} = \frac{u_*}{\kappa z} \left(1 - 15 \frac{z}{L} \right)^{-1/4} \quad \text{Unstable}$$

$$= \frac{u_*}{\kappa z} \left(1 + 4.7 \frac{z}{L} \right) \quad \text{Stable}$$

Buoyant plume for a point source

$$\frac{d}{dz} \pi R^2 w = 2\pi R u_e \quad (\text{i})$$

$$\frac{d}{dz} \rho \pi R^2 w = \rho_a 2\pi R u_e \quad (\text{ii})$$

$$\frac{d}{dz} \rho \pi R^2 w^2 = g(\rho_a - \rho) \pi R^2 \quad (\text{iii})$$

(i) and (iii) give

$$\pi R^2 w \left(\frac{\rho_a - \rho}{\rho_a} \right) g = \text{constant} = F_0 \text{ (buoyancy flux)}$$

$$u_e = \alpha w \quad \alpha - \text{this Entrainment coefficient}$$

$$N^2 = -\frac{g}{\rho} \frac{d\rho}{dz} = \frac{g}{T} \frac{dT}{dz}$$

$$\text{Actually} \quad \frac{g}{T} \left(\frac{dT}{dz} - \frac{dT}{dz} \Big|_{\text{DALR}} \right)$$

N = Brunt – Vaisala Frequency or Buoyancy Frequency

4A8: Environmental Fluid Mechanics

Part II: Dispersion of Pollution in the Atmospheric Environment

DATA CARD

Transport equation for the mean of the reactive scalar ϕ :

$$\frac{\partial \bar{\phi}}{\partial t} + \bar{u}_j \frac{\partial \bar{\phi}}{\partial x_j} = \frac{\partial}{\partial x_j} \left(K \frac{\partial \bar{\phi}}{\partial x_j} \right) + \bar{w}$$

Transport equation for the variance of the reactive scalar ϕ :

$$\frac{\partial g}{\partial t} + \bar{u}_j \frac{\partial g}{\partial x_j} = \frac{\partial}{\partial x_j} \left(K \frac{\partial g}{\partial x_j} \right) + 2K \left(\frac{\partial \bar{\phi}}{\partial x_j} \right)^2 - \frac{2}{T_{turb}} g + 2\overline{\phi'w'}$$

Mean concentration of pollutant after instantaneous release of Q kg at $t=0$:

$$\bar{\phi}(x, y, z, t) = \frac{Q}{8(\pi t)^{3/2} (K_x K_y K_z)^{1/2}} \exp \left[-\frac{1}{4t} \left(\frac{(x-x_0)^2}{K_x} + \frac{(y-y_0)^2}{K_y} + \frac{(z-z_0)^2}{K_z} \right) \right]$$

Gaussian plume spreading in two dimensions from a source at $(0,0,z_0)$ emitting Q kg/s:

$$\bar{\phi}(x, y, z) = \frac{Q}{2\pi U \sigma_y \sigma_z} \exp \left[-\left(\frac{y^2}{2\sigma_y^2} + \frac{(z-z_0)^2}{2\sigma_z^2} \right) \right]$$

One-dimensional spreading from line source emitting Q/L kg/s/m:

$$\bar{\phi}(x, y) = \frac{Q}{UL} \frac{1}{\sqrt{2\pi}\sigma_y} \exp \left(-\frac{y^2}{2\sigma_y^2} \right)$$

Relationship between eddy diffusivity and dispersion coefficient:

$$\sigma^2 = 2 \frac{x}{U} K$$

Engineering Part IIB 2008

4A8 – Environmental Fluid Mechanics

Numerical Answers

DR E MASTORAKOS

Q1: -

Q2: (b) 9.51 m; 0.126 m/s; 0.019 N/m^2 ; -18 W/m^2
(c) $3.47 \times 10^{-3} \text{ m}^2/\text{s}^3$; $-5.26 \times 10^{-4} \text{ m}^2/\text{s}^3$.

Q3: -

Q4: (b) 60 s.

2008 IIB 4A8 ENVIRONMENTAL FLUID MECHANICS DRE MASTORAKOS

- i) a) The flows created when the pressure forces are balanced by Coriolis forces, are called Geostrophic flow. They are governed by

$$\frac{1}{\rho} \nabla P = -2 \underline{\Omega} \times \underline{u}$$

where P includes the centrifugal forces also

The boundary layer between a geostrophic flow and a solid boundary with no-slip condition is called Ekman layer.

Within the Ekman layer, the velocity vector changes its direction slowly to align with the geostrophic velocity at the end of the layer. The tip of the velocity vectors describes a spiral in space, which is called Ekman Spiral.

In the Ekman layer, the balance is between Coriolis & viscous forces.

ii) Brunt-Väisälä frequency is the natural frequency of oscillation (or resonant frequency), which occurs when a fluid parcel is displaced in a stratified flow with stable stratification.

(iii) Features of turbulence:

Unsteady, Three-dimensional in nature, random & irregular, involves a wide range of Scales, highly diffusive and dissipative, and rotational. It obeys the laws of Continuum.

Closure Problem:

In the statistical methods of analysis of turbulence, governing equations for mean quantities involve correlations of fluctuating quantities. These correlations are unknown.

If governing equations for the correlations are derived, it will involve high order correlations which are also unknown. The appearance of correlations cannot be avoided unless modelling approximations are used. This is called turbulence closure problem.

(iv) TKE - In the data sheet.

Sources: 1) mean wind shear

2) Buoyancy - unstable temperature gradient.

3) Topography - waves

Sinks: 1) viscous dissipation
2) Buoyancy - stable Temperature (or) density stratification.

b) $PV = RT$, hydrostatics: $\frac{dp}{dz} = -\rho g$

I & IInd laws of Thermodynamics.

$$TdS = dh - vdp$$

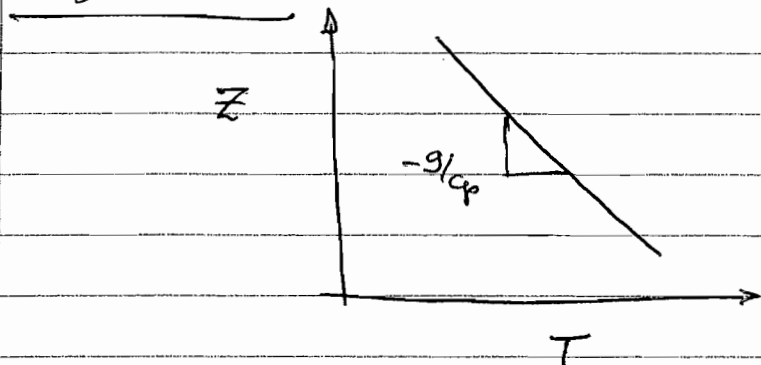
reversible adiabatic process $\Rightarrow dS = 0$

$$dh = + v dp \Rightarrow \frac{dp}{dz} = + \frac{c_p}{v} \frac{dT}{dz}$$

$$-\rho g = + \rho c_p \frac{dT}{dz}$$

$$\Rightarrow \boxed{\frac{dT}{dz} = -g/c_p}$$

Significance:



As $z \uparrow$ $P \downarrow \Rightarrow \rho \downarrow$
But $T \downarrow$ as $z \uparrow$
 $\Rightarrow \rho \uparrow$

$\Rightarrow$ Drop in ρ because of decrease in P is

Compensated by the decrease in T .

$\Rightarrow$ a fluid parcel displaced by Δz , stays there.
"Neutrally stable"

2)

$$\frac{dv}{dz} = \frac{u_*}{kz} (1 + 4.7 \frac{z}{L}) \quad k = 0.4$$

$$a) \int_0^v dv = \frac{u_*}{k} \int_{z_0}^z \left(\frac{1}{z} + \frac{4.7}{L} \right) dz$$

$$v - v_0 = \frac{u_*}{k} \left[\ln \frac{z}{z_0} + \frac{4.7}{L} (z - z_0) \right]$$

$$v_0 = 0 \Rightarrow \boxed{v = \frac{u_*}{k} \left[\ln \left(\frac{z}{z_0} \right) + \frac{4.7}{L} (z - z_0) \right]}$$

b)

$$2 = \frac{u_*}{k} \left[\ln 50 + \frac{4.7}{L} \times 4.9 \right] \quad \text{--- (1)}$$

$$3 = \frac{u_*}{k} \left[\ln 100 + 4.7 \times \frac{9.9}{L} \right] \quad \text{--- (2)}$$

$$\frac{(2)}{(1)} \Rightarrow 1.5 = \frac{4.61L + 46.53}{3.91L + 23.03} \Rightarrow \boxed{L = 9.51 \text{ m}}$$

from (2)

$$u_* = 3 \times 0.4 \left[\ln 100 + \frac{46.53}{L} \right]^{-1}$$

$$\Rightarrow \boxed{u_* = 0.126 \text{ m/sec}}$$

Shear stress at the surface $\tau_w = \rho U_*^2$
(in the data sheet)

$$\tau_w = 1.2 * (0.126)^2 = 0.0191 \text{ N/m}^2$$

$$\tau_w = 0.0191 \text{ N/m}^2$$

Surface heat flux' (Q)

from data sheet $L = - \frac{U_*^3}{k g/T Q/scp}$

$$\Rightarrow Q = - \frac{U_*^3 \rho c_p T}{k g L} = - 0.018 \text{ kW/m}^2$$

$$Q = -18 \text{ W/m}^2$$

c) TKE production via mean shear

$$- \overline{u_i' u_k'} \frac{\partial \overline{u_i}}{\partial x_k} \quad \text{from data sheet.}$$

$$\approx U_*^2 \left(\frac{dU}{dz} \right)_{z=5}$$

$$= (0.126)^2 * \frac{0.126}{0.4 * 5} \left(1 + 4.7 \frac{5}{9.51} \right) = 3.47 \times 10^{-3}$$

$$\text{Production via mean shear} = \underline{\underline{3.47 \times 10^{-3} \text{ m}^2/\text{s}^3}}$$

(6)

via Buoyancy mechanism:

$$\frac{-gQ}{\rho C_p T} \quad (\text{from the data sheet})$$

$$= \frac{9.81 \times 18 \times 10^{-3}}{1.2 \times 1 \times 280} = -5.26 \times 10^{-4} \text{ m}^2/\text{s}^3.$$

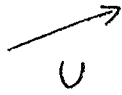
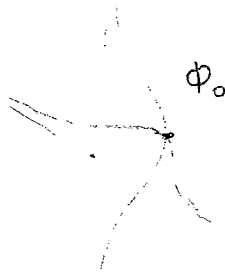
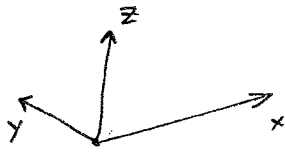
$$d) \frac{\text{Thermal}}{\text{mechanical}} = \frac{-gQ/\rho C_p T}{u_*^2 \left(\frac{d\bar{u}}{dz} \right)}$$

$$u_*^2 \left(\frac{d\bar{u}}{dz} \right) = \frac{u_*^3}{k} \frac{4.7}{L} \quad \text{But } L \approx \frac{-u_*^3 \rho C_p T}{k g Q}$$

$$= \frac{-u_*^3}{k} \frac{4.7 \times g Q}{u_*^3 \rho C_p T} = -4.7 \frac{g Q}{\rho C_p T}$$

$$\Rightarrow \frac{\text{thermal dissipation}}{\text{mechanical production}} = \frac{1}{4.7} = \underline{\underline{0.213}}$$

③



At $y=z=0$, $\bar{\phi} = \phi_0 = \frac{Q}{2\pi U \sigma^2} \exp\left[-\frac{0}{2\sigma^2} + \frac{0}{2\sigma^2}\right]$

$$\Rightarrow \phi_0 = \frac{Q}{2\pi U \sigma^2} \quad (\text{equal } \sigma)$$

- (a) Close to the surface, we expect the vertical lengthscale to be limited. The strong shear of the velocity profile also implies turbulence generation of the streamwise component of the velocity fluctuations.

Hence we expect $\sigma_y > \sigma_z$ for neutral stability.

(For strongly unstable situations, buoyancy production in the vertical direction makes $\sigma_z > \sigma_y$ in general)

- (b) Transport equation for the fluctuations includes production term $\left[2K\left(\frac{\partial \bar{\phi}}{\partial x_i}\right)^2\right]$, which by neglecting gradients in x-direction is $2K\left[\left(\frac{\partial \bar{\phi}}{\partial y}\right)^2 + \left(\frac{\partial \bar{\phi}}{\partial z}\right)^2\right]$

Dissipation term is $\frac{2}{T_{turb}} \bar{\phi}$.

From Gaussian plume equation

$$\bar{\phi} = \frac{Q}{2\pi U \sigma^2} \exp\left(-\frac{y^2}{2\sigma^2}\right) \cdot \exp\left(-\frac{z^2}{2\sigma^2}\right)$$

we get

$$\frac{\partial \bar{\Phi}}{\partial y} = \frac{Q}{2\pi} \frac{1}{U} \frac{1}{\sigma^2} \left(-\frac{2y}{2\sigma^2} \right) \exp\left(-\frac{y^2}{2\sigma^2}\right) \exp\left(-\frac{z^2}{2\sigma^2}\right)$$

$$\Rightarrow \left(\frac{\partial \bar{\Phi}}{\partial y} \right)^2_{y=z=\sigma} = \Phi_0^2 \frac{\sigma^2}{\sigma^4} \exp(-1) \exp(-1) = \Phi_0^2 \frac{1}{\sigma^2} e^{-2}$$

and similarly for $\left(\frac{\partial \bar{\Phi}}{\partial z} \right)^2_{y=z=\sigma}$

$$\Rightarrow \text{Production term} = 2K \cdot \frac{\Phi_0^2}{\sigma^2} e^{-2} + 2K \frac{\Phi_0^2}{\sigma^2} e^{-2}$$

$$= 4 \frac{\Phi_0^2 K}{\sigma^2} e^{-2}$$

Dissipation = Production implies

$$\frac{2g}{T_{\text{turb}}} = 4 \frac{\Phi_0^2 K}{\sigma^2} e^{-2}$$

$$\Rightarrow \frac{g}{\Phi_0^2} = \frac{2K T_{\text{turb}}}{\sigma^2} e^{-2}$$

$$\Rightarrow \frac{g^{1/2}}{\Phi_0} = \frac{(2K T_{\text{turb}})^{1/2}}{\sigma} e^{-1} \quad \text{QED.}$$

(c) The correct turbulent lengthscale to use here ($\sigma \sim x^1$) is σ . Hence $K \equiv C \cdot u' \sigma$ and since $T_{\text{turb}} = \frac{\sigma}{u'}$,

$$\frac{g^{1/2}}{\Phi_0} = \frac{(2 C \cdot u' \sigma \frac{\sigma}{u'})^{1/2}}{\sigma} e^{-1}$$

$$\Rightarrow \text{for } \frac{g^{1/2}}{\Phi_0} = 0.25, \quad C = 0.23$$

$$\Rightarrow \boxed{K = 0.23 u' \sigma}$$

4. (a) Mass balance for pollutant in street canyon.

$$\left(\text{rate of accumulation} \right) = \left(\text{rate of emission} \right) - \left(\text{rate of efflux from top} \right)$$

$$\frac{d}{dt} (c \cdot HWL) = Q \cdot WL - \bar{u}' \cdot WL$$

$$H \frac{dc}{dt} = Q - u(c-b)$$

$$\Rightarrow \boxed{\frac{dc}{dt} = \frac{Q}{H} - \frac{u}{H} (c-b)} \quad \text{QED}$$

(b) At long times (steady-state), $\frac{dc}{dt} = 0$

$$\Rightarrow c = b + \frac{Q}{u} \Rightarrow c = 2 \times 10^{-8} \text{ kg/m}^3$$

(note: this is not needed)

At $t=0$, $Q=0$

$$\Rightarrow \frac{dc}{dt} = -\frac{u}{H} c$$

$$\Rightarrow c = c_0 \exp\left(-\frac{u}{H} t\right)$$

$$\text{For } \frac{c}{c_0} = 0.05, \frac{u}{H} t \approx 3 \Rightarrow t = \frac{3H}{u}$$

$$\Rightarrow \boxed{t = 60 \text{ s}}$$

(C). The exchange velocity u may be affected by:

- wind speed U (high U implies high turbulent intensity $\Rightarrow$ high u)
- hot street ($\Rightarrow$ buoyant production of turbulence)
- Upwind street canyons or obstacles, which may affect local velocity profile $\Rightarrow$ shear
- Time of day (night or day), sunlight levels
- Stability of atmosphere