ENGINEERING TRIPOS PART IIB

Tuesday 6 May 2008 9 to 10.30

4A10

FLOW INSTABILITY

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments: 4A10 data sheet (two pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 The analysis of sound reflected from layers of rock in the ocean floor can be used to locate oil and gas reservoirs. The sound field is commonly measured by acoustic 'aerials', which are long cylinders towed behind a ship (Fig.1). The cylinder consists of a stiff outer elastic hose of thickness, d and inner radius a ($d \ll a$), the core being filled with oil of density ρ_1. The oil can be considered to be incompressible and inviscid. A distribution of hydrophones along the centre-line of the cylinder measures the pressure fluctuations.

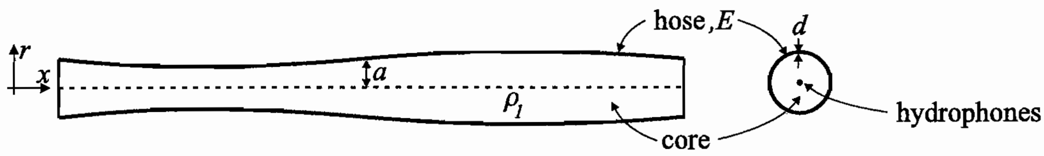

Fig. 1

The hose material has Young's modulus E and its damping is described by δ, where $\delta \ll 1$. For linear axisymmetric perturbations, the pressure difference across the hose is related to its radial displacement, η, by

$$[p]_{r=a}^{r=a+d} = -\frac{Ed}{a^2}(1+\mathrm{i}\delta)\eta \ .$$

(a) For linear axisymmetric disturbances proportional to $e^{\mathrm{i}(\omega t+kx)}$, explain why the velocity potential in the oil is of the form $\phi(r,x,t) = A\,\mathrm{I}_0(kr)\,e^{\mathrm{i}(\omega t+kx)}$, where $\mathrm{I}_0(kr)$ is the modified Bessel function and A is a constant. (All formulae on the data sheet may be used without proof.) [20%]

(b) If the pressure perturbation on the outer surface of the hose is $p_0 e^{\mathrm{i}(\omega t+kx)}$, determine A and hence show that the centre-line pressure perturbation is given by

$$p(0,x,t) = \frac{\rho_1\omega^2}{\rho_1\omega^2\mathrm{I}_0(ka) - (Edk/a^2)(1+\mathrm{i}\delta)\mathrm{I}_0'(ka)}\ e^{\mathrm{i}(\omega t+kx)} \ ,$$

where the dash denotes differentiation with respect to the argument of the function. [50%]

(c) For small ka, the function $\mathrm{I}_0(ka) \approx 1$ and $\mathrm{I}_0'(ka) \approx ka/2$. Show that the centre-line pressure perturbation then simplifies to

$$p(0,x,t) = \frac{\omega^2}{\omega^2 - c_b^2k^2(1+\mathrm{i}\delta)}\ p_0 e^{\mathrm{i}(\omega t+kx)} \ ,$$

where $c_b = \{Ed/(2\rho_1 a)\}^{0.5}$. Comment on the meaning of this solution when (i) $\omega \gg c_b k$, (ii) $\omega \ll c_b k$ and (iii) $\omega \approx c_b k$. What form of motion would you expect when $\omega = c_b k$? [30%]

2 (a) Derive Rayleigh's criterion for the stability of inviscid incompressible flows with circular streamlines. [30%]

(b) How is the instability affected by viscosity? [5%]

(c) Give two examples of practical systems susceptible to a Rayleigh instability, one example where its onset is advantageous and one where its onset is disadvantageous. [5%]

(d) Use Rayleigh's criterion to determine the stability of the flow

(i) between two co-axial cylinders rotating in the same direction with the same angular velocity; [15%]

(ii) external to a single cylinder rotating in a very large region of fluid; [15%]

(iii) with angular velocity Ω such that:

$$\Omega = \begin{cases} 0 & \text{for} \quad 0 \leq r \leq R_1 \,, \\ A + \frac{B}{r^2} & \text{for} \quad R_1 \leq r \leq R_2 \,, \\ \Omega_0 \frac{R_2^2}{r^2} & \text{for} \quad R_2 \leq r \,, \end{cases}$$

where the constants A and B are such that Ω is continuous. [30%]

3 (a) A helium jet discharges into air from a contoured nozzle that has a loudspeaker upstream. When the loudspeaker is switched off, the jet has a natural self-excited bulging oscillation at 983 Hz, which is directly analogous to vortex shedding behind a bluff body. A diagram of the apparatus and an image of the jet are shown in Fig. 2.

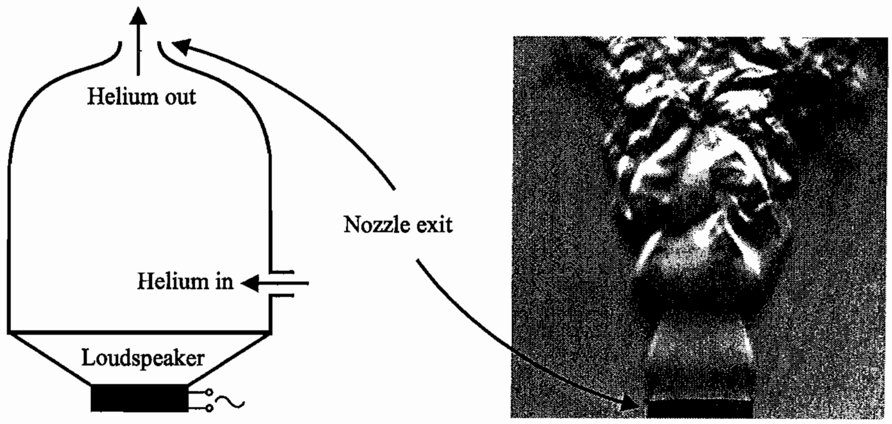

Fig. 2

The loudspeaker is fed a sinusoidal signal at 1023 Hz, whose amplitude is measured in terms of the root mean squared milli-voltage (mV_{rms}). The amplitude is increased from 200 mV_{rms} to 900 mV_{rms} in steps of 100 mV_{rms}. The jet's velocity is measured 1 jet diameter downstream of the nozzle exit. Its power spectral density (PSD) is shown in Fig. 3.

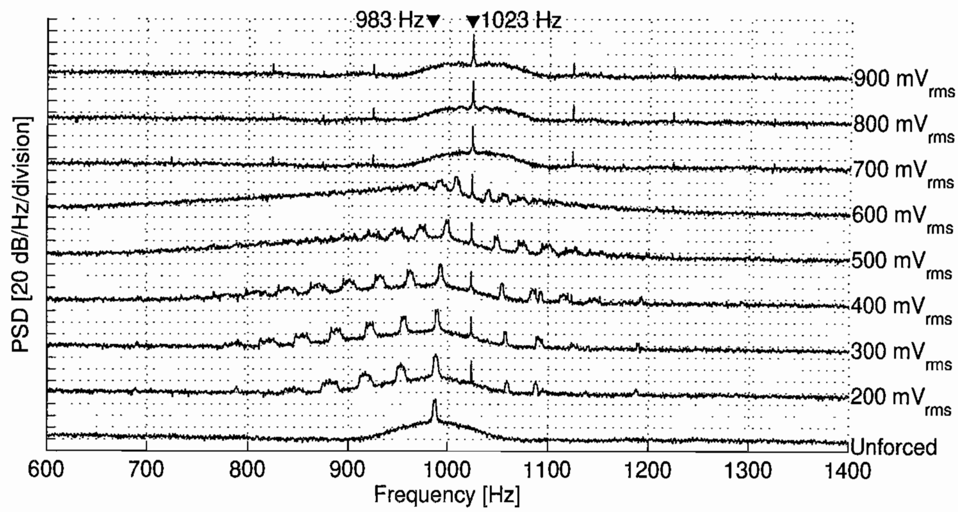

Fig. 3

(cont.

(i) Describe the features of the PSD plots in Fig. 2 and relate them to the behaviour of the jet. [30%]

(ii) What would you expect to see if the loudspeaker were forced at 1000 Hz and, in another experiment, at 1050 Hz? [10%]

(b) An under-sea walkway is proposed at a tourist resort in water of depth 30 m. It has circular cross-section and it will be suspended half way between the sea floor and the surface. Its diameter is 3 m and its mass per unit length is 885 kg m^{-1}. Its resonant frequency of vertical motion in air is 0.5 Hz. The velocity of water in that region never exceeds 3 m s^{-1} in any direction. Are there any problems with the design from a fluid-structure interaction perspective? [60%]

Some of the following information may be useful. The density of seawater is 1027 kg m^{-3} and its viscosity is 1.2×10^{-3} kg m^{-1} s^{-1}. The density of air is 1.2 kg m^{-3} and its viscosity is 1.8×10^{-5} kg m^{-1} s^{-1}. The added mass coefficient for a circular cross-section is 1. The Strouhal number of vortex shedding is defined as $St = fD/U$, where f is the frequency in Hz, D is the diameter of the circular tunnel and U is the speed of the flow. For shedding behind a cylinder at a Reynolds number $Re > 1000$ the Strouhal number is 0.2.

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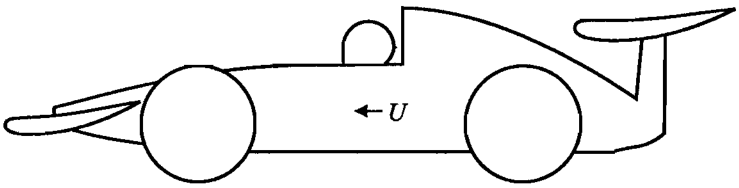


Fig. 4

The fluid-structure interaction of the racing car in Fig. 4 is to be modelled by the mass-spring-aerofoil system shown in Fig. 5. At a given velocity, U, the aerofoils produce downforces F_1 and F_2, which, together with the mass of the car, m, are balanced by the suspension springs, which are distance l apart. All forces and displacements are measured around these equilibrium values. The vertical displacement of the car, y, is measured at the elastic axis, which is situated half way between the suspension springs.

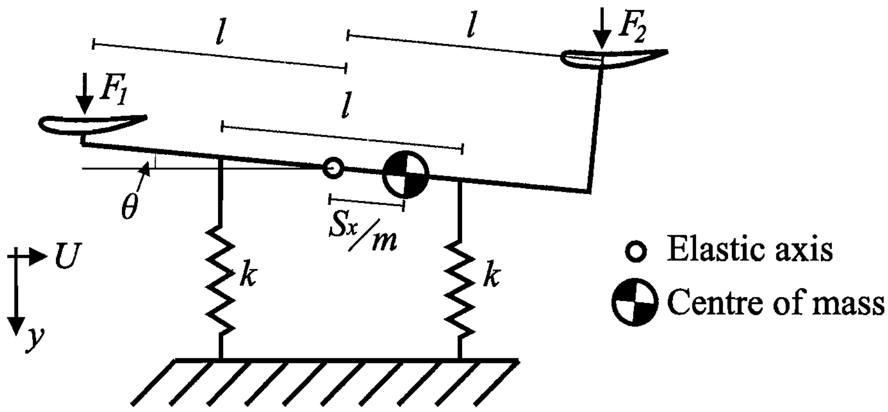


Fig. 5

(a) Evaluate the apparent angles of attack of the front and rear aerofoils, α_1 and α_2, in terms of θ, l, U, $\dot{y}$ and $\dot{\theta}$. [20%]

(b) The aerofoils have surface areas A_1 and A_2 and the same lift coefficient, C_L, where $\partial C_L/\partial\alpha$ is constant and positive. The flow velocity, U, is sufficiently large that $\dot{y}/U$ and $L\dot{\theta}/U$ can be neglected in this question. Find expressions for F_1 and F_2 in terms of $\rho, U, A_1, A_2, \theta$ and $\partial C_L/\partial\alpha$. [10%]

(cont.

The two suspension springs, each with stiffness k, are modelled as a translational spring with stiffness $k_y = 2k$ and a torsional spring with torsional stiffness $k_\theta = kl^2/2$. The translational spring and the torsional spring act around the elastic axis. Similarly, the aerodynamic forces, F_1 and F_2, are modelled as a translational force, $F_y = F_1 + F_2$, and a moment about the elastic axis, $F_\theta = l \times (F_2 - F_1)$. The centre of mass is a distance S_x/m behind the elastic axis. Ignoring damping, the translational and torsional equations of motion are

$$\begin{aligned} m\ddot{y} + S_x\ddot{\theta} + k_y y &= F_y \ , \\ I_\theta\ddot{\theta} + S_x\ddot{y} + k_\theta\theta &= F_\theta \ , \end{aligned}$$

where I_θ is the car's moment of inertia about the elastic axis.

(c) Derive two expressions for the conditions at which the car will be unstable and describe these unstable motions. You may assume that $mI_\theta > S_x^2$. [50%]

(d) Without detailed calculations, briefly discuss some aspects of the car's stability in terms of its design parameters. What would be the effect of the suspension dampers on this instability? [20%]

END OF PAPER

STABILITY OF PARALLEL SHEAR FLOW

Rayleigh's inflexion point theorem

A parallel shear flow with profile $U(z)$ is only unstable to inviscid perturbations if

$$\frac{d^2U}{dz^2}=0 \quad \text{for some } z.$$

CONVECTIVE FLOW

The Boussinesq approximation leads to

$$\nabla\cdot u=0$$

$$\frac{Du}{Dt}=-\frac{1}{\rho_0}\nabla p+(1-\alpha(T-T_0))g+\upsilon\nabla^2 u$$

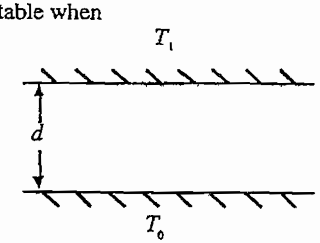
and

$$\frac{DT}{Dt}=\kappa\nabla^2 T$$

Rayleigh-Bénard convection

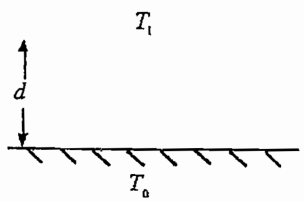
A fluid between two **rigid** plates is unstable when

$$\text{Ra}\geq 1708$$



A liquid with a **free** upper surface is unstable when

$$\frac{\text{Ra}}{\text{Ra}_c}+\frac{\text{Ma}}{\text{Ma}_c}\geq 1$$



where

$$\text{Ra}=\frac{g\alpha(T_0-T_1)d^3}{\upsilon\kappa},\quad \text{Ma}=\frac{\chi(T_0-T_1)d}{\rho\upsilon\kappa} \quad \text{with } \chi=-\frac{d\sigma}{dT}$$

$\text{Ra}_c\approx 670$ $\quad$ $\text{Ma}_c\approx 80.$

USEFUL MATHEMATICAL FORMULA

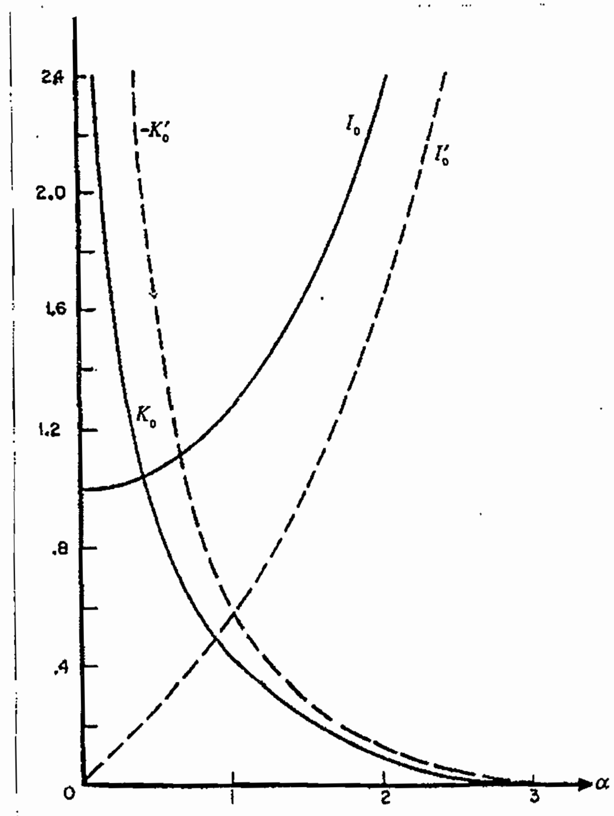
Modified Bessel equation

$I_0(kr)$ and $K_0(kr)$ are two independent solutions of

$$\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-k^2f=0.$$

$I_0(kr)$ is finite at $r=0$ and tends to infinity as $r\to\infty$,

$K_0(kr)$ is infinite at $r=0$ and tends to zero as $r\to\infty$.



$I_0(\alpha)$, $K_0(\alpha)$, $I_0'(\alpha)$, $K_0'(\alpha)$
where $'$ denotes a derivative

Module 4A10 Data Card

EQUATIONS OF MOTION

For an incompressible isothermal viscous fluid:

Continuity $\nabla \cdot \boldsymbol{u} = 0$

Navier Stokes $\rho \dfrac{D\boldsymbol{u}}{Dt} = -\nabla p + \mu \nabla^2 \boldsymbol{u}$

D/Dt denotes the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

IRROTATIONAL FLOW $\nabla \times \boldsymbol{u} = 0$

velocity potential ϕ,

$$\boldsymbol{u} = \nabla\phi \quad \text{and} \quad \nabla^2\phi = 0$$

Bernoulli's equation

for inviscid flow $\dfrac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + gz + \dfrac{\partial\phi}{\partial t} =$ constant throughout flow field.

KINEMATIC CONDITION AT A MATERIAL INTERFACE

A surface $z = \eta(x,y,t)$ moves with fluid of velocity $\boldsymbol{u} = (u,v,w)$ if

$$w = \frac{D\eta}{Dt} = \frac{\partial\eta}{\partial t} + \boldsymbol{u} \cdot \nabla\eta \quad \text{on } z = \eta(x,t).$$

For η small and $\boldsymbol{u}$ linearly disturbed from $(U,0,0)$

$$w = \frac{\partial\eta}{\partial t} + U\frac{\partial\eta}{\partial x} \quad \text{on } z = 0.$$

SURFACE TENSION σ AT A LIQUID-AIR INTERFACE

Potential energy

The potential energy of a surface of area A is σA.

Pressure difference

The difference in pressure Δp across a liquid-air surface with principal radii of curvature R_1 and R_2 is

$$\Delta p = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

For a surface which is almost a circular cylinder with axis in the x-direction, $r = a + \eta(x,\theta,t)$ (η is very small so that η^2 is negligible)

$$\Delta p = \frac{\sigma}{a} + \sigma\left(-\frac{\eta}{a^2} - \frac{\partial^2\eta}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2\eta}{\partial\theta^2}\right),$$

where Δp is the difference between the internal and the external surface pressure.

For a surface which is almost plane with $z = \eta(x,t)$ (η is very small so that η^2 is negligible)

$$\Delta p = -\sigma\frac{\partial^2\eta}{\partial x^2}$$

where Δp is the difference between pressure at $z = \eta^+$ and $z = \eta^-$.

ROTATING FLOW

In steady flows with circular streamlines in which the fluid velocity and pressure are functions of radius r only:

Rayleigh's criterion

The flow is unstable / stable to inviscid axisymmetric disturbances if Γ^2 decreases / increases with r.

$\Gamma = 2\pi r V(r)$ is the circulation around a circle of radius r.

Navier Stokes equation simplifies to

$$0 = \mu\left(\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{V}{r^2}\right)$$

$$-\rho\frac{V^2}{r} = -\frac{dp}{dr}.$$