ENGINEERING TRIPOS PART IIB

Friday 2 May 2008 9 to 10.30

Module 4A12

TURBULENCE

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments: 4A12 Data Card
(i) Vortex Dynamics (1 page)
(ii) Turbulence (2 pages)

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) State Helmholtz's two laws of vortex dynamics. Prove the first law by noting that a short line element in the fluid, $\delta\,\mathbf{l}$, which always consists of the same fluid particles, evolves according to

$$\frac{D}{Dt}(\delta\,\mathbf{l}) = \left((\delta\,\mathbf{l})\cdot\nabla\right)\mathbf{u}\ .$$

[30%]

(b) A vorticity field, $\boldsymbol{\omega}(\mathbf{x},t)$, consists of two, thin, closed vortex tubes whose centrelines are C_1 and C_2 and vorticity fluxes are Φ_1 and Φ_2. The net helicity of the flow associated with the tubes is defined as $H = \int \mathbf{u}\cdot\boldsymbol{\omega}\,dV$.

(i) Confirm that the contribution to H from tube 1 is given by the line integral

$$H_1 = \oint_{C_1} \mathbf{u}\cdot(\Phi_1 d\mathbf{r})\ .$$

[25%]

(ii) Use Stokes' theorem to show that $H = 0$ if the tubes are not linked (Fig. 1a), that $H = 2\Phi_1\Phi_2$ if the tubes are linked in a right-handed fashion (Fig. 1b), and that $H = -2\Phi_1\Phi_2$ if the tubes are linked in a left-handed fashion (Fig. 1c). [25%]

(c) If the flow is inviscid, it may be shown that H is conserved. Explain this using Helmholtz's laws. [20%]

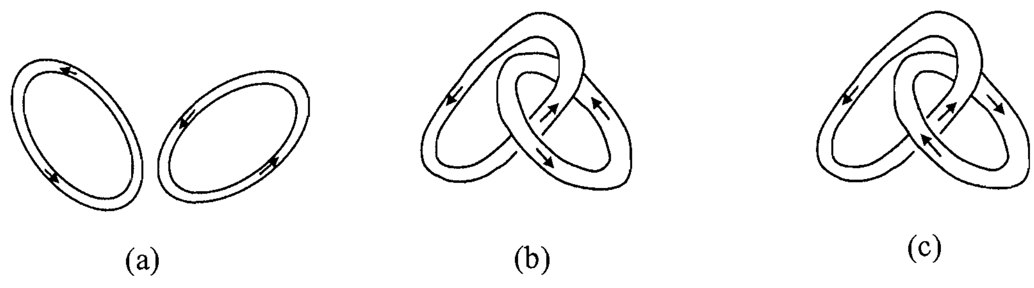


Fig. 1

2 (a) A two-dimensional vortex tube is aligned with the z-axis and has the vorticity distribution

$$\boldsymbol{\omega} = \frac{\Gamma_0}{\pi \delta^2} \exp\left(-\frac{r^2}{\delta^2} \right) \hat{\mathbf{e}}_z \ .$$

Here, Γ_0 is the net flux along the vortex tube and δ is its characteristic radius. The associated velocity field is, in polar coordinates,

$$\mathbf{u} = \frac{\Gamma_0}{2\pi r} \left[1 - \exp\left(-\frac{r^2}{\delta^2} \right) \right] \hat{\mathbf{e}}_\theta \ .$$

Show that this represents an exact solution of the two-dimensional vorticity equation

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \nu \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \omega}{\partial r} \right)$$

provided $\delta = (Ct)^{1/2}$ for some constant C. Find the relationship between C and ν. [60%]

(b) The peak vorticity, $\Gamma_0 / (\pi \delta^2)$, decays with time while the characteristic radius, δ, increases. Give a physical explanation for this. [20%]

(c) Sketch Burgers vortex. Why does the peak vorticity and radius of the Burgers vortex stay constant, in contrast to the example above? [20%]

3 (a) Derive a relationship between the Reynolds numbers based on the Taylor lengthscale, $\mathrm{Re}_\lambda = u\lambda / \nu$, and the integral lengthscale, $\mathrm{Re}_t = uL_{turb} / \nu$, where u is the characteristic large-scale velocity and ν the kinematic viscosity. Derive also a relationship for the ratio λ / η_K, where η_K is the Kolmogorov lengthscale. [50%]

(b) In a steady planar jet flowing in the x-direction, at a distance $x = L$ from the origin, the flow has characteristic width δ in the y-direction. On the axis, the mean velocity is U_c and the characteristic turbulent velocity is u_0. Give the order of magnitude of the terms in the mean streamwise momentum equation, and hence show that

$$\frac{u_0}{U_c} = \left(\frac{\delta}{L}\right)^{1/2} .$$

You may neglect viscous stresses and you can use the result from the y-momentum equation that

$$-\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} = \frac{\partial \overline{v'v'}}{\partial x} .$$

[50%]

4 Consider freely decaying homogeneous isotropic turbulence in a wind tunnel, created by passing a uniform flow through a grid. In a moving frame of reference, the turbulent kinetic energy k is given by, for $t > t_0$,

$$\frac{k}{k_0} = \left(\frac{t}{t_0}\right)^{-m}$$

where t_0 is an arbitrary time at which the kinetic energy is k_0, and m is a constant. At this time, the large-eddy turbulent lengthscale L and the eddy turn-over time $T = L / k^{1/2}$ are L_0 and T_0 respectively.

(a) By reference to the usual production mechanisms of turbulence, discuss if it is reasonable to assume that wind tunnel turbulence is homogeneous and isotropic very close and very far from the grid. [30%]

(b) Experimental data give that $m = 1.2$. Find the rate of growth of L and the rate of growth of T. [50%]

(c) If the turbulent Reynolds number $\mathrm{Re}_t = k^{1/2} L / \nu$ is 100 at $t = t_0$, find the time when the Reynolds number decays to unity. [20%]

END OF PAPER

Vortex Dynamics Data Card

Grad, Div and Curl in Cartesian Coordinates

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Integral Theorems

Gauss : $\int (\nabla \cdot A) dV = \oint A \cdot dS$

Stokes : $\int (\nabla \times A) \cdot dS = \oint A \cdot dl$

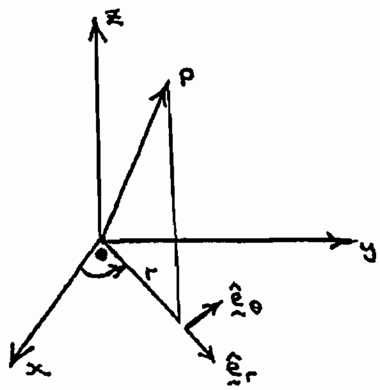
Vector Identities

$\nabla(A \cdot B) = (A \cdot \nabla)B + (B.\nabla)A + A \times (\nabla \times B) + B \times (\nabla \times A)$
$\nabla \cdot (fA) = f(\nabla \cdot A) + A \cdot \nabla f$
$\nabla \times (A \times B) = A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla)A - (A \cdot \nabla)B$
$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$
$\nabla \times (\nabla f) = 0$
$\nabla \cdot (\nabla \times A) = 0$

Cylindrical Coordinates (r, θ, z)

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot A = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$



$$\nabla \times A = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$

$$\nabla \times A = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)$$

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4A12: Turbulence

Data Card

Assume incompressible fluid with constant properties.

Continuity:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

Mean momentum:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \partial^2 \overline{u}_i / \partial x_j^2 - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \overline{g}_i$$

Mean scalar:

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_i \frac{\partial \overline{\phi}}{\partial x_i} = D \frac{\partial^2 \overline{\phi}}{\partial x_i^2} - \frac{\partial \overline{u_i' \phi'}}{\partial x_i}$$

Turbulent kinetic energy ($k = \overline{u_i' u_i'}/2$):

$$\begin{aligned} \frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} &= -\frac{1}{\rho}\frac{\partial \overline{u_j' p'}}{\partial x_j} - \frac{1}{2}\frac{\partial \overline{u_j' u_i' u_i'}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2} \\ & -\overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{g_i' u_i'} \end{aligned}$$

Scalar fluctuations ($\sigma^2 = \overline{\phi' \phi'}$):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_j \frac{\partial \sigma^2}{\partial x_j} = D \frac{\partial^2 \sigma^2}{\partial x_j^2} - 2\overline{\phi' u_j' \frac{\partial \phi'}{\partial x_j}} - 2\overline{\phi' u_j'} \frac{\partial \overline{\phi}}{\partial x_j} - 2D \overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2}$$

Energy dissipation:

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \approx \frac{u^3}{L_{turb}}$$

Scalar dissipation:

$$2\overline{N} = 2D\overline{\left(\frac{\partial\phi'}{\partial x_j}\right)^2} \approx 2\frac{\varepsilon}{k}\sigma^2$$

Scaling rule for shear flow, flow dominant in direction x_1:

$$\frac{u}{L_{turb}} \sim \frac{\partial\overline{u}_1}{\partial x_2}$$

Kolmogorov scales:

$$\begin{aligned}\eta_K &= \left(\nu^3/\varepsilon\right)^{1/4}\\ \tau_K &= \left(\nu/\varepsilon\right)^{1/2}\\ v_K &= \left(\nu\varepsilon\right)^{1/4}\end{aligned}$$

Taylor microscale:

$$\varepsilon = 15\nu\frac{u^2}{\lambda^2}$$

Eddy viscosity (general):

$$\begin{aligned}\overline{u_i'u_j'} &= -\nu_{turb}\left(\frac{\partial\overline{u}_i}{\partial x_j} + \frac{\partial\overline{u}_j}{\partial x_i}\right) + \frac{2}{3}k\delta_{ij}\\ \overline{u_j'\phi'} &= -D_{turb}\frac{\partial\overline{\phi}}{\partial x_j}\end{aligned}$$

Eddy viscosity (for simple shear):

$$\overline{u_1'u_2'} = -\nu_{turb}\frac{\partial\overline{u}_1}{\partial x_2}$$