ENGINEERING TRIPOS PART IIB

Monday 21 April 2008 9 to 10.30

Module 4A14

SILENT AIRCRAFT INITIATIVE

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment
Data sheet for 4A14 (two pages)

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) Explain why a rigid-walled annular duct of inner radius a and outer radius b can be approximated as a rectangular duct when $(b-a)/a \ll 1$. [10%]

(b) What are the appropriate boundary conditions for acoustic waves in the rectangular duct? [10%]

(c) Show that for acoustic disturbances of frequency ω these boundary conditions lead to axial wave numbers

$$k_{mn} = \sqrt{\frac{\omega^2}{c_0^2} - \left(\frac{m\pi}{b-a}\right)^2 - \left(\frac{n}{R}\right)^2} \quad \text{for integer } m \text{ and } n.$$ [60%]

c_0 is the speed of sound and $R = \frac{1}{2}(a+b)$.

(d) Hence show that in the annular duct all modes of circumferential mode number n are 'cut-off' at radian frequencies ω less than nc_0/R. [10%]

(e) What condition on fan speed is required if rotor-alone noise is to be cut-off? [10%]

2 Cabin bleed air is expelled at a mass flow rate $m_1(t)$ through a small hole into a plenum of volume V. The air is finally discharged underneath the wing at a mass flow rate $m_2(t)$ through a short duct as shown in Fig. 1. The mass flow rate $m_2(t)$ is related to the plenum pressure p_2 by a constant κ such that

$$\frac{dm_2}{dt} = \kappa p_2'(t).$$

A dash denotes perturbations from the mean and c_0 is the speed of sound.

(a) Show that for fluctuations of radian frequency ω

$$m_2'(t) = \frac{m_1'(t)}{1 - V\omega^2/(\kappa c_0^2)}$$ [30%]

(b) Write down the relationship between dm_2/dt and the far-field sound pressure at distance r away from the position O, where the flow exhausts through the wing. [20%]

(c) Hence, for $V = 4.8 \times 10^{-4}$ m^3, $\kappa = 0.04$ m, $r = 200$ m, $c_0 = 343$ ms^{-1}, determine the Sound Pressure Level (SPL) if there is an rms modulation of 2×10^{-4} kg s^{-1} in the bleed flow rate $m_1(t)$ at a frequency of 500Hz. [40%]

(d) If fluctuations in the mass flow $m_1(t)$ are expected at 500Hz, what plenum volume V would you choose? [10%]

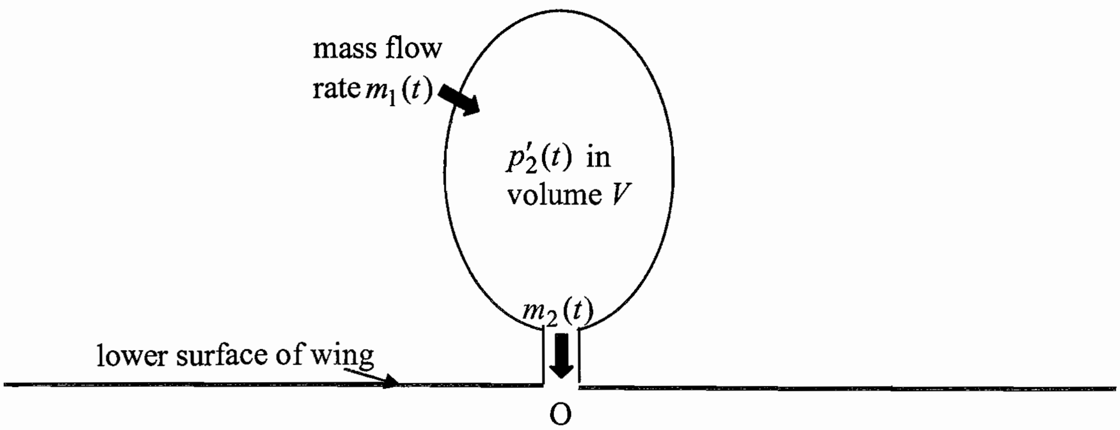


Fig. 1

3 (a) (i) Consider the linear sound speed profile

$$c_0(x) = \alpha x + \beta,$$

where α and β are constants with $\beta > 0$. Use Snell's Law to show that the rays lie on arcs of a circle, whose centre and radius are to be found in terms of α, β and the angle θ_0 the ray makes with the x-axis at $x = 0$. [30%]

Comment on the difference between the cases of α positive and α negative. [5%]

(ii) Now consider the sound speed profile

$$\begin{aligned} c_0(x) &= \alpha(x+H) \quad \text{for} \quad 0 \le x \le H \\ &= 2\alpha H \quad \text{for} \quad x \ge H, \end{aligned}$$

where H is a positive constant. Sketch a diagram showing the fate of rays launched from the origin, and identify the critical angle to the x axis which divides the two possible types of behaviour. [20%]

(b) A plane sound wave with acoustic pressure

$$p' = p_i \exp(i\omega t - ik_0 y),$$

where k_0 positive, is incident on the rigid plate $\{-\infty < x < 0, y = 0\}$.
Calculate the scattered field at

(i) $x = -L, \quad y = h$
(ii) $x = -L, \quad y = -h,$

where h and L are positive constants, with $k_0 h \gg 1, \quad k_0 L \gg 1$. [35%]

Sketch a graph of the variation of acoustic pressure with x along the line $y = h$. [10%]

4 (a) Derive Lighthill's eighth-power law for jet noise. Formulae on the data card may be used without proof. [50%]

(b) A plane sound wave is normally incident on a wall which is free to vibrate. Show that the ratio of the amplitude of the transmitted pressure to that of the incident pressure is

$$\frac{2\rho_0 c_0}{\sqrt{4\rho_0^2 c_0^2 + m^2\omega^2}} ,$$

where ρ_0 is the mean air density, c_0 is the sound speed, m is the mass per unit area of the wall and ω is the frequency. [40%]

Comment on the physical significance of the ratio

$$\frac{m\omega}{\rho_0 c_0}$$

for the attenuation of sound by a wall. [10%]

END OF PAPER

Module 4A14 SILENT AIRCRAFT INITIATIVE – DATA CARD

USEFUL DATA AND DEFINITIONS

Physical Properties

Speed of sound in an ideal gas $\sqrt{\gamma RT}$, where T is temperature in Kelvins

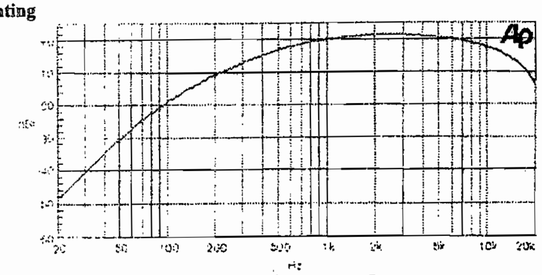
Units of sound measurement

SPL (sound pressure level) $= 20\log_{10}\left(\dfrac{p'_{rms}}{2.10^{-5}\,\text{Nm}^{-2}}\right)$ dB

IL (intensity level) $= 10\log_{10}\left(\dfrac{\text{intensity}}{10^{-12}\,\text{watts m}^{-2}}\right)$ dB

PWL (power level) $= 10\log_{10}\left(\dfrac{\text{sound power output}}{10^{-12}\,\text{watts}}\right)$ dB

A-weighting



SPL mode A-Weighting Curve

Definitions

Surface impedance , Z_s, relates the pressure perturbation applied to a surface, p' , to its normal velocity v_n; $p' = Z_s v_n$.

Characteristic impedance of a fluid $\rho_0 c_0$

Nondimensional surface impedance of a surface $Z_s/\rho_0 c_0$

Transmission loss = $10\log_{10}\left(\dfrac{\text{incident sound power}}{\text{transmitted sound power}}\right)$

Absorption coefficient of a sound absorber $= \dfrac{\text{sound power absorbed}}{\text{incident sound power}}$

Wavelength, λ, for sound waves with angular frequency ω, $\lambda = 2\pi c_0/\omega$

Wave-number $k_0 = \omega / c_0 = 2\pi / \lambda$

Phase speed = ω/k

Group velocity = $\dfrac{\partial \omega}{\partial k}$

Helmholtz number (or compactness ratio) = $k_0 D$, where D is a typical dimension of the source.

Strouhal number = $\omega D/(2\pi U)$ for sound of frequency ω, produced in a flow with speed U, length scale D.

BASIC EQUATIONS FOR LINEAR ACOUSTICS

Conservation of mass $\dfrac{\partial \rho'}{\partial t} + \rho_0\, \nabla\cdot \mathbf{v}' = 0$

Conservation of momentum $\rho_0 \dfrac{\partial \mathbf{v}'}{\partial t} + \nabla p' = 0$

Isentropic $c_0^2 = \left.\dfrac{dp}{d\rho}\right|_S$

These equations combine to give the **wave equation** $\dfrac{1}{c_0^2}\dfrac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = 0$

Energy density $e = \frac{1}{2}\rho_0 v'^2 + \frac{1}{2} p'^2 / \rho_0 c_0^2$

Intensity $\mathbf{I} = p'\mathbf{v}'$

$\text{div}\bar{\mathbf{I}} = 0$ for statistically stationary (in time) sound fields.

Velocity potential $\phi'(\mathbf{x},t)$ satisfies the wave equation and $p' = -\rho_0 \dfrac{\partial \phi'}{\partial t}$, $\mathbf{v} = \nabla\phi'$.

SIMPLE WAVE FIELDS

1D or plane wave

The general solution of the 1D wave equation is $p'(x,t) = f(t - x/c_0) + g(t + x/c_0)$, where f and g are arbitrary functions. In a plane wave propagating to the right $p' = \rho_0 c_0 u'$; in a plane wave propagating to the left $p' = -\rho_0 c_0 u'$, u' being the particle velocity.

Spherically symmetric sound fields

The general spherically symmetric solution of the 3D wave equation is $\phi'(r,t) = \dfrac{f(t - r/c_0)}{r} + \dfrac{g(t + r/c_0)}{r}$

where r is the distance from the source; f and g are arbitrary functions.

cos θ dependence

The general solution of the 3D wave equation with cos θ dependence is

$$p'(\mathbf{x},t) = \frac{\partial}{\partial x}\left[\frac{f(t - r/c_0)}{r} + \frac{g(t + r/c_0)}{r}\right] = \cos\theta \frac{\partial}{\partial r}\left[\frac{f(t - r/c_0)}{r} + \frac{g(t + r/c_0)}{r}\right].$$

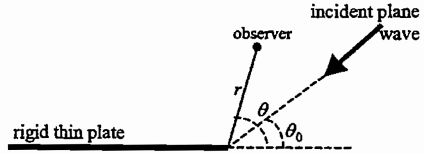
In a cylindrical duct, radius a_0

$p'(\mathbf{x},t) = e^{i(\omega t + n\theta)} J_n(z_{mn} r/a_0)(Ae^{-ikx_3} + Be^{ikx_3})$, where z_{mn} is the mth zero of $\dot{J}_n(z)$ and $k = (k_0^2 - z_{mn}^2 / a_0^2)^{1/2}$.

SCATTERING

For an incident plane wave of amplitude p_i, propagating at an angle θ_0, the diffracted pressure a distance r from a sharp edge is

$$p_i\left(\frac{2}{\pi k_0 r}\right)^{1/2} \frac{\sin(\theta_0/2)\sin(\theta/2)}{\cos\theta_0 + \cos\theta} \exp(-ik_0 r - i\pi/4)$$



USEFUL MATHEMATICAL FORMULAE

In spherical polar coordinates (r,θ,ϕ)

$$\nabla p' = \left(\frac{\partial p'}{\partial r}, \frac{1}{r}\frac{\partial p'}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial p'}{\partial \phi}\right)$$

For $\mathbf{v}' = (v_r', v_\theta', v_\phi')$, $\quad \nabla\cdot\mathbf{v}' = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r') + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta') + \frac{1}{r\sin\theta}\frac{\partial v_\phi'}{\partial\phi}.$

$$\nabla^2 p' = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p'}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial p'}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 p'}{\partial\phi^2}.$$

In cylindrical polar coordinates (r,θ,x_3)

$$\nabla p' = \left(\frac{\partial p'}{\partial r}, \frac{1}{r}\frac{\partial p'}{\partial \theta}, \frac{\partial p'}{\partial x_3}\right)$$

For $\mathbf{v}' = (v_r', v_\theta', v_3')$, $\quad \nabla\cdot\mathbf{v}' = \frac{1}{r}\frac{\partial}{\partial r}(r v_r') + \frac{1}{r}\frac{\partial v_\theta'}{\partial\theta} + \frac{\partial v_3'}{\partial x_3}$, $\quad \nabla^2 p' = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p'}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p'}{\partial\theta^2} + \frac{\partial^2 p'}{\partial x_3^2}.$

Heaviside function $H(t-\tau) = 1 \;\; t>\tau \;;\; = 0 \;\; t<\tau$

δ-functions

Kronecker delta $\delta_{ij} = 1$ if $i=j$; 0 if $i \neq j$

1D δ-function: $\delta(t) = 0$ for $t \neq 0$; $f(t)\delta(t-\tau) = f(\tau)\delta(t-\tau)$

$$\int_{-\infty}^{\infty}\delta(t)dt = 1 \text{ and } \int_{-\infty}^{\infty}\delta(t-\tau)f(t)dt = f(\tau) \text{ for any function } f(t).$$

3D δ-function: $\delta(\mathbf{x}) = \delta(x_1)\,\delta(x_2)\,\delta(x_3)$; $\delta(\mathbf{x}) = 0$ for $|\mathbf{x}| \neq 0$; $f(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) = f(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})$

$\int_V \delta(\mathbf{x})dV = 1$ for any volume V that includes the origin

and

$\int_V \delta(\mathbf{x}-\mathbf{y})f(\mathbf{x})dV = f(\mathbf{x})$ for any function $f(\mathbf{x})$ and volume V that includes $\mathbf{x}$.

$$\nabla^2\left(\frac{1}{|\mathbf{x}|}\right) = -4\pi\delta(\mathbf{x}).$$

Autocorrelation

$F(\xi)$, the autocorrelation of $f(y) = \overline{f(y)f(y+\xi)}$

$$F(0) = \overline{f^2}$$

Integral lengthscale ℓ $\quad \ell\overline{f^2} = \int_{-\infty}^{\infty}F(\xi)d\xi.$

SOURCES

Point sources

monopole of strength $Q(t)$ at the origin generates a pressure field

$$p'(\mathbf{x},t) = \frac{Q(t-|\mathbf{x}|/c_0)}{4\pi|\mathbf{x}|}.$$

dipole of strength $\mathbf{F}(t)$ at the origin generates a pressure field

$$p'(\mathbf{x},t) = -\frac{\partial}{\partial x_i}\left[\frac{F_i(t-|\mathbf{x}|/c_0)}{4\pi|\mathbf{x}|}\right] = \frac{x_i}{4\pi}\left[\frac{1}{|\mathbf{x}|^3}F_i(t-|\mathbf{x}|/c_0) + \frac{1}{|\mathbf{x}|^2 c_0}\frac{\partial F_i}{\partial t}(t-|\mathbf{x}|/c_0)\right]$$

Distributed sources

Monopole, strength $q(\mathbf{x},t)$, wave equation $\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)p' = q$, pressure field $p'(\mathbf{x},t) = \int\frac{q(\mathbf{y},t-|\mathbf{x}-\mathbf{y}|/c_0)}{4\pi|\mathbf{x}-\mathbf{y}|}dV$

Dipole, strength $\mathbf{f}(\mathbf{x},t)$, wave equation $\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)p' = -\nabla\cdot\mathbf{f}$, $\quad p'(\mathbf{x},t) = -\frac{\partial}{\partial x_i}\int\frac{f_i(\mathbf{y},t-|\mathbf{x}-\mathbf{y}|/c_0)}{4\pi|\mathbf{x}-\mathbf{y}|}dV$

Quadrupole, strength $T_{ij}(\mathbf{x},t)$, equation $\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)p' = \frac{\partial T_{ij}}{\partial x_i\partial x_j}$, $\quad p'(\mathbf{x},t) = \frac{\partial^2}{\partial x_i\partial x_j}\int\frac{T_{ij}(\mathbf{y},t-|\mathbf{x}-\mathbf{y}|/c_0)}{4\pi|\mathbf{x}-\mathbf{y}|}dV$

Far-field form $|\mathbf{x}| >> |\mathbf{y}|$, $\mathbf{y}$ near origin

$$|\mathbf{x}-\mathbf{y}| \sim |\mathbf{x}| - \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|} + O(|\mathbf{x}|^{-2}),\; \frac{1}{|\mathbf{x}-\mathbf{y}|} \sim \frac{1}{|\mathbf{x}|} + O(|\mathbf{x}|^{-2}),\; \frac{\partial}{\partial x_i} \sim -\frac{x_i}{|\mathbf{x}|c_0}\frac{\partial}{\partial t} + 0(|\mathbf{x}|^{-1}).$$

Physical sources

Lighthill's theory shows that jet noise is generated by quadrupoles of strength $T_{ij} = \rho v_i v_j + (p' - c_0^2\rho')\delta_{ij} - \tau_{ij}$.

The Ffowcs-Williams Hawkings equation shows that foreign bodies in linear motion generate far-field sound

$$p'(\mathbf{x},t) = \frac{1}{4\pi|\mathbf{x}|}\frac{\partial}{\partial t}\int_S \rho_0 d\mathbf{S}\bullet\mathbf{v}\left(\mathbf{y}, t - \frac{|\mathbf{x}|}{c_0} + \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|c_0}\right) + \frac{x_i}{4\pi|\mathbf{x}|^2 c_0}\frac{\partial}{\partial t}\int_S dS_i\, p\left(\mathbf{y}, t - \frac{|\mathbf{x}|}{c_0} + \frac{\mathbf{x}\cdot\mathbf{y}}{|\mathbf{x}|c_0}\right)$$

SOUND POWER

Sound power from a source, $P = \int_S \bar{\mathbf{I}}\bullet d\mathbf{S} = \int_{S_\infty}\frac{\overline{p'^2}}{\rho_0 c_0}dS$ for a statistically stationary source.

For a spherically symmetrical sound field $P = \frac{\overline{p'^2}}{\rho_0 c_0}4\pi r^2$ where p' is the pressure at radius r.

For a sound field, which is a function of spherical polar coordinates r, θ only, and independent of ϕ,

$$P = 2\pi r^2\int_0^{\pi}\frac{\overline{p'^2}}{\rho_0 c_0}\sin\theta\, d\theta$$