ENGINEERING TRIPOS PART IIB

Friday 25 April 2008 2.30 to 4

Module 4C8

APPLICATIONS OF DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
4C8 datasheet (3 pages)

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 A quarter-car model is shown in Fig. 1. The sprung mass is m_s and the unsprung mass is m_u. Their corresponding displacements are z_s and z_u. The vertical tyre stiffness is k_t and the suspension stiffness and damping are k and c. The road input displacement is z_r.

(a) Show that

$$m_s \ddot{z}_s(t) + m_u \ddot{z}_u(t) - k_t\left[z_r(t) - z_u(t)\right] = 0 .$$ [20%]

(b) Transfer functions relating road input velocity to body acceleration, working space and tyre force outputs are defined as

$$H_{BA}(s) = \frac{\ddot{z}_s(s)}{\dot{z}_r(s)}; \quad H_{WS}(s) = \frac{z_s(s) - z_u(s)}{\dot{z}_r(s)}; \quad H_{TF}(s) = \frac{k_t\left[z_r(s) - z_u(s)\right]}{\dot{z}_r(s)},$$

where s is the Laplace transform variable. Explain why road input velocity (as opposed to displacement) is an appropriate input quantity. Briefly state the relevance of each of the output quantities. [20%]

(c) Use some or all of the expressions given in (a) and (b) to show that

$$-m_s H_{BA} + \left(1 - \frac{m_u}{k_t}\omega^2\right) H_{TF} = j\omega\, m_u .$$ [35%]

(d) Why is the expression given in (c) relevant to understanding the trade-off (typically presented in the form of a 'conflict diagram') between root-mean-square values of body acceleration and tyre force? [10%]

(e) From the expression given in (c) find an expression for the frequency at which H_{BA} is independent of H_{TF}, and state H_{BA} at this frequency. Comment on the result. [15%]

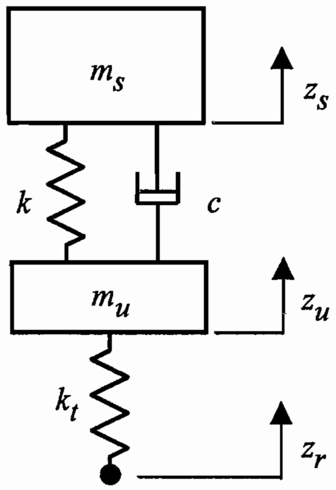

Fig. 1

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2 Figure 2 shows a side view of an idealized model of a tandem-axle suspension of a heavy truck. The two closely-spaced axles, each of point mass m_u, are connected by a light rigid beam of length $2a$. Springs of stiffness k_t represent the vertical tyre stiffnesses. The suspension, with stiffness k and damping c, is connected by a frictionless pivot to the centre of the beam. The centre of the beam has vertical displacement z_u and pitch angle θ_u. The upper end of the suspension is connected to the sprung mass m_s which moves in the vertical direction with displacement z_s. Inputs are the road displacements at the leading and trailing axles, z_1 and z_2. The vehicle travels with speed U in the direction shown, hence the road input displacement z_2 is a delayed version of the displacement input z_1. Parameter values are: m_s = 10,000 kg; m_u = 300 kg; k = 500 kN m^{-1}; k_t = 2.67 MN m^{-1}; c = 40 kN s m^{-1}; and a = 1 m.

(a) One of the three natural modes of vibration of the undamped system has eigenvector $\{ z_s \; z_u \; \theta_u \}^T = \{ 0 \; 0 \; 1 \}^T$. Sketch this mode shape and hence determine the natural frequency of this mode. Note that it is not necessary to derive the full equations of motion. [10%]

(b) The remaining two natural modes of undamped vibration involve z_s and z_u only, with θ_u = 0. Sketch the approximate shape of these two modes of vibration and hence *estimate* the natural frequency of each mode. Note that it is not necessary to derive the full equations of motion. (Hint – the mode shapes are the same as those of a quarter-car model obtained by setting a = 0). [20%]

(c) Wheelbase filtering is the mechanism by which the speed U and wheelbase $2a$ influence the excitation of the modes of vibration of the vehicle.

(i) Consider the vehicle travelling very slowly over a sinusoidal road profile of wavelength λ and displacement amplitude Z_r. Let the amplitudes of the displacements z_u and θ_u be Z_u and Θ_u respectively. Sketch the magnitudes of the 'bounce gain' Z_u/Z_r and 'pitch gain' $\Theta_u a/Z_r$ of the wheelbase filter as functions of the spatial frequency $2a/\lambda$. [40%]

(cont.

(ii) The vehicle travels on a randomly rough road at a speed of $U = 20 \text{ m s}^{-1}$. Are there any frequencies within the range 0 Hz to 25 Hz at which the response might be problematical? How could the suspension be modified to improve the vibration performance? [30%]

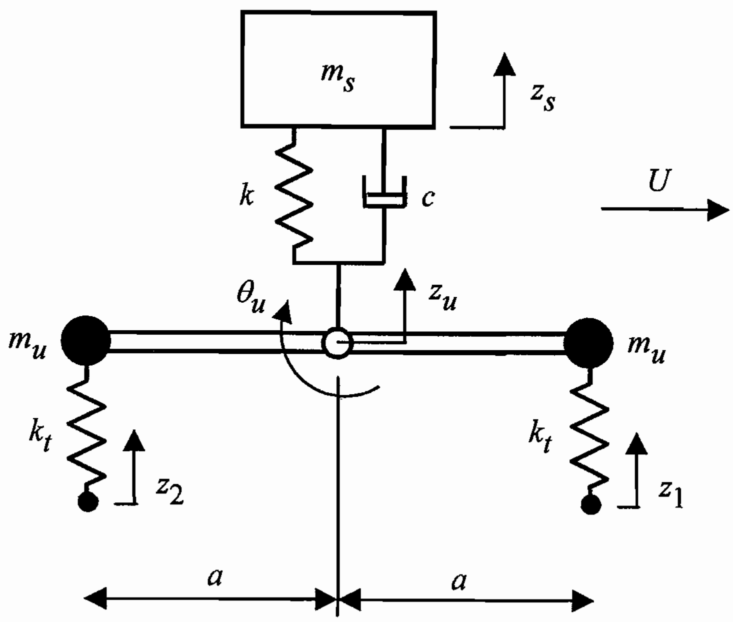

Fig. 2

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3 A space probe is put into a circular Earth orbit of radius r_1 by a booster rocket. Once in orbit, the booster is explosively jettisoned. The impulses of this explosion act tangentially along the orbit path, such that the velocity of the probe increases and that of the booster decreases (without becoming negative). After the explosion the booster and the probe, which both have the same mass, separate with a relative velocity V. The Earth may be treated as a perfect sphere.

(a) Draw a sketch showing the original circular orbit and the two subsequent elliptical orbits, in relation to the centre of the Earth. [20%]

(b) Find the major-axis lengths of the two new orbits, in terms of r_1, μ and V, where μ is as defined on the data sheet. [40%]

(c) What must the value of V be (in terms of r_1 and μ) for the probe just to escape the Earth's gravitational field? [20%]

(d) What value of V is needed such that the perigee of the booster's new orbit lies on the surface of the Earth, allowing the booster to coast back to Earth? [20%]

4 (a) Explain clearly the terms *mean anomaly*, *mean motion*, *eccentric anomaly*, and *true anomaly*, using diagrams where necessary. Derive an expression that relates the mean anomaly to the eccentric anomaly. [40%]

(b) A GPS receiver has calculated that the mean anomaly of a particular GPS satellite is 0.6315 radians at a given instant. The satellite has also broadcast the following data about its ephemeris:

eccentricity = 0.020;
semi-major axis = 26610 km;
right ascension = 30.0°;
inclination = 55.0°;
argument of perigee = 45.0°;
other corrections are negligible.

(i) Show that the eccentric anomaly of the satellite is 0.6435 radians, and thus calculate its true anomaly. [20%]

(ii) If the receiver is situated at the North Pole, calculate the distance between the receiver and the satellite, to the nearest kilometre. You may assume that the Earth is a perfect sphere of radius 6378 km. [40%]

END OF PAPER

Engineering Tripos Part IIB

Data sheet for Module 4C8: Applications of Dynamics

DATA ON VEHICLE VIBRATION

Random Vibration

$$E\left[x(t)^2\right] = \frac{1}{T}\int_{t=0}^{t=T} x(t)^2\,dt = \int_{\omega=-\infty}^{\omega=\infty} S_x(\omega)\,d\omega \qquad \left(\text{or } \int_{\omega=0}^{\omega=\infty} S_x(\omega)\,d\omega \quad \text{if } S_x(\omega) \text{ is single sided}\right)$$

$$S_{\dot{x}}(\omega) = \omega^2 S_x(\omega)$$

Single Input – Single Output

$$S_y(\omega) = \left|H_{yx}(\omega)\right|^2 S_x(\omega)$$

$$y(\omega) = H_{yx}(\omega)x(\omega)$$

Two Input – Two Output

$$\begin{Bmatrix} y_1(\omega) \\ y_2(\omega) \end{Bmatrix} = \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix} \begin{Bmatrix} x_1(\omega) \\ x_2(\omega) \end{Bmatrix}$$

$$\begin{bmatrix} S_{11}^y(\omega) & S_{12}^y(\omega) \\ S_{21}^y(\omega) & S_{22}^y(\omega) \end{bmatrix} = \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix}^* \begin{bmatrix} S_{11}^x(\omega) & S_{12}^x(\omega) \\ S_{21}^x(\omega) & S_{22}^x(\omega) \end{bmatrix} \begin{bmatrix} H_{11}(\omega) & H_{12}(\omega) \\ H_{21}(\omega) & H_{22}(\omega) \end{bmatrix}^T$$

* means complex conjugate, T means transpose

If x_1 and x_2 are uncorrelated:

$$S_{(x_1+x_2)}(\omega) = S_{x_1}(\omega) + S_{x_2}(\omega)$$

$$S_{12}^x(\omega) = S_{21}^x(\omega) = 0$$

$$E\left[\left(x_1(t) + x_2(t)\right)^2\right] = E\left[x_1(t)^2\right] + E\left[x_2(t)^2\right]$$

DATA ON POTENTIAL THEORY AND ORBITS

1 For a distribution of mass with density $\rho(\mathbf{r})$ the gravitational potential U satisfies Poisson's equation

$$\nabla^2 U = -4\pi G\rho$$

where G is the gravitational constant (= 6.67×10^{-11} Nm^{2}kg^{-2}). The gravitational force $\mathbf{F}$ experienced by a unit mass is given by

$$\mathbf{F} = \nabla U .$$

2 *In vacuo* $\rho = 0$, so that U satisfies Laplace's equation

$$\nabla^2 U = 0 .$$

3 For a point mass M at the origin

$$U(\mathbf{r}) = GM/|\mathbf{r}| .$$

For a general distribution of matter

$$U(\mathbf{r}) = G\iiint \frac{\rho(\mathbf{x})\,d^3\mathbf{x}}{|\mathbf{r}-\mathbf{x}|} .$$

For a thin spherical shell of radius a and mass dM

$$U(r) = \begin{cases} GdM/|r|, & r > a \\ GdM/a, & r < a \end{cases}$$

4 Equations of motion for a particle in a plane orbit, in plane polar coordinates (r, θ):

$$\ddot{r} - r\dot{\theta}^2 = f_r \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = f_\theta$$

where f_r, f_θ are the radial and transverse force components, per unit mass.

If $f_\theta = 0$ (i.e. for a central force) the second equation leads to conservation of angular momentum:

$$r^2\dot{\theta} = h = constant .$$

5 For a central force, the substitution $u = 1/r$ leads to an equation for the *shape* of the orbit, expressed as $u = u(\theta)$. The central force (assumed attractive) is described by a function $f(u)$ per unit mass, and for a given angular momentum per unit mass h the orbit satisfies

$$\frac{\partial^2 u}{\partial\theta^2} + u = \frac{f(u)}{h^2u^2} .$$

6 The equation of an ellipse in polar coordinates (r,θ) relative to a focus is

$$r = \frac{L}{(1+e\cos\theta)}$$ where e is the eccentricity.

The semi-major axis is $a = L/(1-e^2)$, the semi-minor axis is $b = L/\sqrt{1-e^2}$.

7 The mean anomaly M, the eccentric anomaly E and the true anomaly θ are related by

$$M = E - e\sin E \qquad \text{and} \qquad \cos\theta = \frac{\cos E - e}{1 - e\cos E}$$ where e is the eccentricity.

8 Spherical polar coordinates. Define (r,θ,ϕ) so that r is radial distance, θ is angle from the polar axis (co-latitude) and ϕ is the angle of longitude. Then:

$$\nabla U = \frac{\partial U}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\mathbf{e}_\phi$$

$$\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial U}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial U}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}$$

9 Axisymmetric solutions to Laplace's equation arising from separation of variables in spherical polar coordinates are

$$U(r,\theta) = \begin{cases} r^n P_n(\cos\theta) \\ r^{-n-1} P_n(\cos\theta) \end{cases}$$

where P_n is the Legendre polynomial of order n, describing the nth zonal harmonic. The first few Legendre polynomials are as follows:

$$P_0(\xi) = 1 \qquad P_1(\xi) = \xi \qquad P_2(\xi) = (3\xi^2 - 1)/2$$

$$P_3(\xi) = (5\xi^3 - 3\xi)/2 \qquad P_4(\xi) = (35\xi^4 - 30\xi^2 + 3)/8 \ .$$

10 The external potential of the Earth can be expressed as a sum of spherical-harmonic contributions. We consider in detail only the effect of the zonal harmonics, whose contribution can be written in standard form

$$U(r,\theta) = \frac{\mu}{r}\left[1 - \sum_{n=2}^{\infty}(R/r)^n J_n P_n(\cos\theta)\right] .$$

For the Earth, $\mu = G\times M_{Earth} = 398603\ km^3s^{-2}$, mean radius $R = 6378$ km,

$$J_2 = 1082\times10^{-6}, \quad J_3 = -2.55\times10^{-6}, \quad J_4 = -1.65\times10^{-6}$$

Gravitational mass of the sun = 332946μ, gravitational mass of the moon = μ/81.3

Mean radius of Earth's orbit = 1.496×10^8 km, that of moon's orbit = 384400 km.