ENGINEERING TRIPOS PART IIB

Wednesday 23 April 2008 2.30 to 4.00

Module 4C9

CONTINUUM MECHANICS

Answer not more than ***two*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
4C9 datasheet (6 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) (i) Using subscript notation, and not otherwise, prove that

$$\underline{a}\times\underline{b}\bullet\underline{c}\times\underline{d}=(\underline{a}\bullet\underline{c})(\underline{b}\bullet\underline{d})-(\underline{a}\bullet\underline{d})(\underline{b}\bullet\underline{c})$$ [25%]

(ii) Expand the expression $e_{mp}\,e_{nq}\,\varepsilon_{mn,pq}=0$ if

$$e_{ij}=\begin{cases}1 \text{ when } i=1,\ j=2\\ 0 \text{ when } i=j\\ -1 \text{ when } i=2,\ j=1\end{cases}$$ [25%]

(b) A yield criterion f is defined by the relation $f=\bar{\sigma}-\alpha\sigma_{kk}-Y\leq 0$ in which $\bar{\sigma}=\sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}}$ is the effective Cauchy stress, α is a constant and Y is the current flow stress.

(i) Take partial derivatives of the yield criterion to give an expression for all components of the strain rate tensor in terms of a scalar rate parameter $\dot{\lambda}$. [25%]

(ii) Using your answer to (i), together with the Voce flow rule

$$\bar{\sigma}=K\left[1-m\exp(-n\bar{\varepsilon})\right]$$

in which $\bar{\varepsilon}=\sqrt{\frac{2}{3}\varepsilon_{ij}\varepsilon_{ij}}$ is the equivalent strain, and K, m and n are material constants, derive a formula for the scalar rate parameter $\dot{\lambda}$ in terms of the effective stress and its time derivative. [25%]

2 A radial force F acts at one side of a circular hole of radius a in an infinitely large plate as shown in Fig. 1. The stress field associated with the applied force can be assumed to decrease with increasing distance from the hole.

(a) The Flamant solution for the force on the edge of a half space suggests an Airy stress function of the form

$$\phi(\rho,\psi) = -\frac{F\rho\psi}{\pi}\sin\psi \; .$$

Assuming that this applies in this example, write down expressions for the stresses $\sigma_{\rho\rho}, \sigma_{\psi\psi}$ and $\sigma_{\rho\psi}$ at a general point P in the material. [25%]

(b) Using the usual relation for the rotation of second order tensors, i.e. $\sigma'_{\alpha\beta} = a_{\alpha i}\, a_{\beta j}\, \sigma_{ij}$, or otherwise, obtain expressions for the stresses $\sigma_{rr}, \sigma_{\theta\theta}$ and $\sigma_{r\theta}$ at point P in terms of F, ρ, ψ and θ where coordinates (r,θ) are as shown in Fig. 1. [25%]

(c) Verify that around the periphery of the hole these expressions reduce to

$$\sigma_{rr} = \frac{F}{2\pi a}(1-\cos\theta)$$

$$\sigma_{\theta\theta} = \frac{F}{2\pi a}(1+\cos\theta)$$

$$\sigma_{r\theta} = \frac{F}{2\pi a}\sin\theta$$

and explain why this means that additional terms are necessary in the original expression for $\phi(\rho,\psi)$. [25%]

(d) By referring to Table I of the Data Sheet suggest the form that these additional terms might take and indicate how the coefficients associated with each of them could be evaluated. [25%]

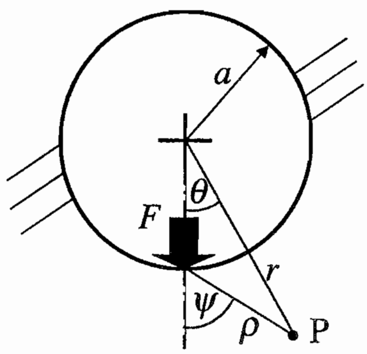

Fig.1

(TURN OVER

3 Figure 2(a) shows a section through an infinite width, symmetric, frictionless extrusion process in which the ram moves at constant speed v. The workpiece is of a rigid-plastic material with yield stress in shear of magnitude k.

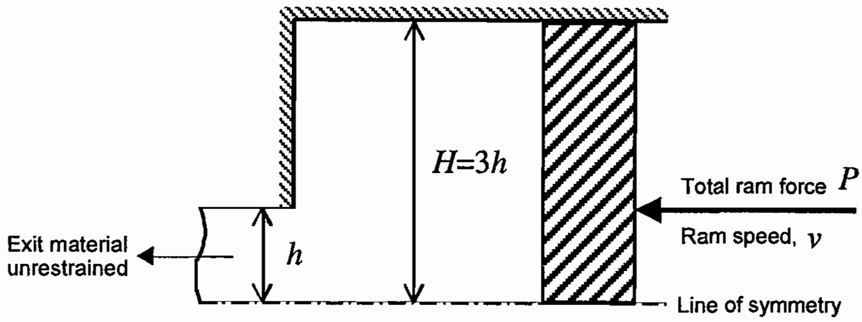

Fig. 2(a)

(a) By defining a suitable statically admissible stress field derive a lower-bound estimate of the ram force. [30%]

(b) By defining a suitable kinematically admissible velocity field derive an upper-bound estimate of the ram force. [30%]

(c) Using the slip-line field shown as ABCD in Fig. 2(b) calculate the ram force using the Henky equations. [30%]

(d) Comment on the applicability of these estimates to a real extrusion process. [10%]

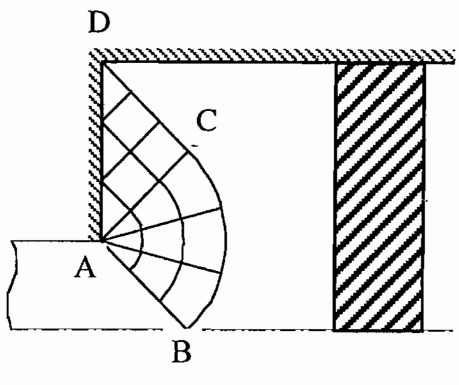

Fig. 2(b)

END OF PAPER

ENGINEERING TRIPOS Part IIB

Module 4C9 Data Sheet

SUBSCRIPT NOTATION

Repeated suffix implies summation

$\underline{a} = a_1\underline{e}_1 + a_2\underline{e}_2 + a_3\underline{e}_3$	$a_i e_i$
$\underline{a} \bullet \underline{b}$	$a_i b_i \equiv a_i b_j \delta_{ij}$
$\underline{c} = \underline{a} \times \underline{b}$	$c_i = e_{ijk} a_j b_k$
$\underline{d} = \underline{a} \times (\underline{b} \times \underline{c})$	$d_k = -e_{ijk} e_{irs} a_j b_r c_s = a_j b_k c_j - a_i b_i c_k$
Kronecker delta δ_{ij}	$\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$
e_{ijk}	e_{ijk} =1 when indices cyclic; = –1 when indices anticyclic and = 0 when any indices repeat
$e-\delta$ identity	$e_{ijk} e_{ilm} \equiv \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$
trace$a$	tr$a = a_{ii} = a_{11} + a_{22} + a_{33}$
$\dfrac{\partial \sigma_{ij}}{\partial x_i} = \dfrac{\partial \sigma_{1j}}{\partial x_1} + \dfrac{\partial \sigma_{2j}}{\partial x_2} + \dfrac{\partial \sigma_{3j}}{\partial x_3}$	$\sigma_{ij,i}$
gradϕ $= \nabla\phi$	$\dfrac{\partial \phi}{\partial x_i} = \phi_{,i}$
div$\underline{V}$	$V_{i,i}$
curl$\underline{V} \equiv \underline{\nabla} \times \underline{V}$	$e_{ijk} V_{k,j}$

<u>Rotation of Orthogonal Axes</u>

If $01'2'3'$ is related to 0123 by rotation matrix a_{ij}

vector v_i becomes	$v'_\alpha = a_{\alpha i} v_i$
tensor σ_{ij} becomes	$\sigma'_{\alpha\beta} = a_{\alpha i} a_{\beta j} \sigma_{ij}$

Evaluation of principal stresses

deviatoric stress $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$

$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$

$I_1 = \sigma_{ii} = \mathrm{tr}\sigma$

$I_2 = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right)$

$I_3 = \frac{1}{6}\left(e_{ijk}e_{pqr}\sigma_{ip}\sigma_{jq}\sigma_{kr}\right)$

$s^3 - J_1 s^2 + J_2 s - J_3 = 0$

$J_1 = s_{ii} = \mathrm{tr}s$; $J_2 = \frac{1}{2}s_{ij}s_{ij}$; $J_3 = \frac{1}{3}s_{ij}s_{jk}s_{ki}$

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equilibrium $\sigma_{ij},_i + b_j = 0$

small strains $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

compatibility $\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{lj,ki} - \varepsilon_{ki,lj} e_{pik}e_{qjl}\varepsilon_{ij,kl} = 0$

equivalent to $e_{pik}e_{qjl}\varepsilon_{ij,kl} \equiv e_{pik}e_{qjl}\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} = 0$

Linear elasticity $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

Hooke's law $E\varepsilon_{ij} = (1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}$

Lamé's equations $\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

von Mises equivalent stress $\bar{\sigma} = \sqrt{\frac{3}{2}s_{ij}s_{ij}} = \sqrt{3J_2}$

equivalent strain increment $\mathrm{d}\bar{\varepsilon} = \sqrt{\frac{2}{3}\mathrm{d}\varepsilon_{ij}\mathrm{d}\varepsilon_{ij}}$

Elastic torsion of prismatic bars

Warping function $\Psi(x_1, x_2)$ satisfies $\nabla^2\Psi = \Psi_{,ii} = 0$

If Prandtl stress function $\phi(x_1, x_2)$ satisfies $\nabla^2\phi = \phi_{,ii} = -2G\alpha$ where α is the twist per unit length then

$$\sigma_{31} = \phi_{,2} = \frac{\partial\phi}{\partial x_2} \ , \ \sigma_{32} = -\phi_{,1} = -\frac{\partial\phi}{\partial x_1} \ \text{and} \ T = 2\iint_A \phi(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2$$

Equivalence of elastic constants

	E	ν	$G{=}\mu$	λ
E, ν	–	–	$\dfrac{E}{2(1+\nu)}$	$\dfrac{\nu E}{(1+\nu)(1-2\nu)}$
E, G	–	$\dfrac{E-2G}{2G}$	–	$\dfrac{(2G-E)G}{E-3G}$
E, λ	–	$\dfrac{E-\lambda+R}{4\lambda}$	$\dfrac{E-3\lambda+R}{4}$	–
ν, G	$2G(1+\nu)$	–	–	$\dfrac{2G\nu}{1-2\nu}$
ν, λ	$\dfrac{\lambda(1+\nu)(1-2\nu)}{\nu}$	–	$\dfrac{\lambda(1-2\nu)}{2\nu}$	–
G, λ	$\dfrac{G(3\lambda+2G)}{\lambda+G}$	$\dfrac{\lambda}{2(\lambda+G)}$	–	–

$R = \sqrt{E^2 + 2E\lambda + 9\lambda^2}$

Two-dimensional Airy Stress function

Biharmonic equation $\nabla^4\phi \equiv \phi,_{\alpha\alpha\beta\beta} = 0$

Stresses $\sigma_{\alpha\beta} = e_{\alpha\gamma}\, e_{\beta\delta}\, \phi,_{\gamma\delta}$

$$\text{where } e_{\alpha\beta} \equiv e_{3\alpha\beta} = \begin{cases} 1 \text{ if } \alpha = 1, \beta = 2 \\ 0 \text{ if } \alpha = \beta \\ -1 \text{ if } \alpha = 2, \beta = 1 \end{cases}$$

Plane stress and plane strain

$$G\varepsilon_{11} = \frac{1}{8}\{\sigma_{11}(1+\kappa) + \sigma_{22}(\kappa - 3)\}$$

$$G\varepsilon_{22} = \frac{1}{8}\{\sigma_{22}(1+\kappa) + \sigma_{11}(\kappa - 3)\}$$

$$G\varepsilon_{12} = \frac{\sigma_{12}}{2}$$

where $\begin{cases} \kappa = (3-\nu)/(1+\nu) & \text{in plane stress and} \\ \kappa = 3 - 4\nu & \text{in plane strain} \end{cases}$

Plasticity

von Mises yield criterion

$$f = \frac{3}{2}\sigma'_{ij}\sigma'_{ij} - Y^2 \equiv \frac{3}{2}s_{ij}s_{ij} - Y^2$$

generalized flow rule

$$\dot{\varepsilon}_{ij} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}}$$

Slip Line Fields

Henky equations

$$p - 2k\phi = \text{constant along } \alpha \text{ line}$$

$$p + 2k\phi = \text{constant along } \beta \text{ line}$$

Geiringer equations

$$\frac{\mathrm{d}v_\alpha}{\mathrm{d}s} = v_\beta \frac{\mathrm{d}\phi}{\mathrm{d}s} \quad \text{along } \alpha \text{ line}$$

$$\frac{\mathrm{d}v_\beta}{\mathrm{d}s} = v_\alpha \frac{\mathrm{d}\phi}{\mathrm{d}s} \quad \text{along } \beta \text{ line}$$

Table I – The Michell solutions — stress components

$\phi(r,\theta)$	σ_{rr}	$\sigma_{\theta\theta}$	$\sigma_{r\theta}$
r^2	2	2	0
$r^2 \ln r$	$2\ln r + 1$	$2\ln r + 3$	0
$\ln r$	$1/r^2$	$-1/r^2$	0
θ	0	0	$1/r^2$
$r^3 \cos\theta$	$2r\cos\theta$	$6r\cos\theta$	$2r\sin\theta$
$r\theta\sin\theta$	$2\cos\theta/r$	0	0
$r\ln r\cos\theta$	$\cos\theta/r$	$\cos\theta/r$	$\sin\theta/r$
$\cos\theta/r$	$-2\cos\theta/r^3$	$2\cos\theta/r^3$	$-2\sin\theta/r^3$
$r^3 \sin\theta$	$2r\sin\theta$	$6r\sin\theta$	$-2r\cos\theta$
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\ln r\sin\theta$	$\sin\theta/r$	$\sin\theta/r$	$-\cos\theta/r$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\sin\theta/r^3$	$2\cos\theta/r^3$
$r^{n+2}\cos n\theta$	$-(n+1)(n-2)r^n\cos n\theta$	$(n+1)(n+2)r^n\cos n\theta$	$n(n+1)r^n\sin n\theta$
$r^{-n+2}\cos n\theta$	$-(n+2)(n-1)r^{-n}\cos n\theta$	$(n-1)(n-2)r^{-n}\cos n\theta$	$-n(n-1)r^{-n}\sin n\theta$
$r^n\cos n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$
$r^{-n}\cos n\theta$	$-n(n+1)r^{-n-2}\cos n\theta$	$n(n+1)r^{-n-2}\cos n\theta$	$-n(n+1)r^{-n-2}\sin n\theta$
$r^{n+2}\sin n\theta$	$-(n+1)(n-2)r^n\sin n\theta$	$(n+1)(n+2)r^n\sin n\theta$	$-n(n+1)r^n\cos n\theta$
$r^{-n+2}\sin n\theta$	$-(n+2)(n-1)r^{-n}\sin n\theta$	$(n-1)(n-2)r^{-n}\sin n\theta$	$n(n-1)r^{-n}\cos n\theta$
$r^n\sin n\theta$	$-n(n-1)r^{n-2}\sin n\theta$	$n(n-1)r^{n-2}\sin n\theta$	$-n(n-1)r^{n-2}\cos n\theta$
$r^{-n}\sin n\theta$	$-n(n+1)r^{-n-2}\sin n\theta$	$n(n+1)r^{-n-2}\sin n\theta$	$n(n+1)r^{-n-2}\cos n\theta$

Table II – The Michell solutions — displacement components

For plane strain $\kappa = 3 - 4\nu$; for planes stress $\kappa = (3-\nu)/(1+\nu)$

$\phi(r,\theta)$	$2Gu_r$	$2Gu_\theta$
r^2	$(\kappa-1)r$	0
$r^2 \ln r$	$(\kappa-1)r\ln r - r$	$(\kappa+1)r\theta$
$\ln r$	$-1/r$	0
θ	0	$-1/r$
$r^3 \cos\theta$	$(\kappa-2)r^2\cos\theta$	$(\kappa+2)r^2\sin\theta$
$r\theta\sin\theta$	$0.5[(\kappa-1)\theta\sin\theta - \cos\theta + (\kappa+1)\ln r\cos\theta]$	$0.5[(\kappa-1)\theta\cos\theta - \sin\theta - (\kappa+1)\ln r\sin\theta]$
$r\ln r\cos\theta$	$0.5[(\kappa+1)\theta\sin\theta - \cos\theta + (\kappa-1)\ln r\cos\theta]$	$0.5[(\kappa+1)\theta\cos\theta - \sin\theta - (\kappa-1)\ln r\sin\theta]$
$\cos\theta / r$	$\cos\theta / r^2$	$\sin\theta / r^2$
$r^3 \sin\theta$	$(\kappa-2)r^2\sin\theta$	$-(\kappa-2)r^2\cos\theta$
$r\theta\cos\theta$	$0.5[(\kappa-1)\theta\cos\theta + \sin\theta - (\kappa+1)\ln r\sin\theta]$	$0.5[-(\kappa-1)\theta\sin\theta - \cos\theta - (\kappa+1)\ln r\cos\theta]$
$r\ln r\sin\theta$	$0.5[-(\kappa+1)\theta\cos\theta - \sin\theta + (\kappa-1)\ln r\sin\theta]$	$0.5[(\kappa+1)\theta\sin\theta + \cos\theta + (\kappa-1)\ln r\cos\theta]$
$\sin\theta / r$	$\sin\theta / r^2$	$-\cos\theta / r^2$
$r^{n+2}\cos n\theta$	$(\kappa-n-1)r^{n+1}\cos n\theta$	$(\kappa+n+1)r^{n+1}\sin n\theta$
$r^{-n+2}\cos n\theta$	$(\kappa+n-1)r^{-n+1}\cos n\theta$	$-(\kappa-n+1)r^{-n+1}\sin n\theta$
$r^n\cos n\theta$	$-nr^{n-1}\cos n\theta$	$nr^{n-1}\sin n\theta$
$r^{-n}\cos n\theta$	$nr^{-n-1}\cos n\theta$	$nr^{-n-1}\sin n\theta$
$r^{n+2}\sin n\theta$	$(\kappa-n-1)r^{n+1}\sin n\theta$	$-(\kappa+n+1)r^{n+1}\cos n\theta$
$r^{-n+2}\sin n\theta$	$(\kappa+n-1)r^{-n+1}\sin n\theta$	$(\kappa-n+1)r^{-n+1}\cos n\theta$
$r^n\sin n\theta$	$-nr^{n-1}\sin n\theta$	$-nr^{n-1}\cos n\theta$
$r^{-n}\sin n\theta$	$nr^{-n-1}\sin n\theta$	$-nr^{-n-1}\cos n\theta$

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