ENGINEERING TRIPOS PART IIB

Wednesday 30 April 2008 2.30 to 4

Module 4F2

ROBUST MULTIVARIABLE CONTROL

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 Consider the mass-spring system shown in Fig. 1, where x is the position of the mass and u is an external force. Assume the nominal values of $M = 0.1$ kg, $\lambda = 1.1$ N·sec/m and $k = 1$ N/m.

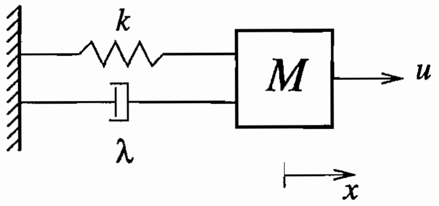

Fig. 1

(a) Show that the system in Fig. 1 can be represented as in Fig. 2 . [10%]

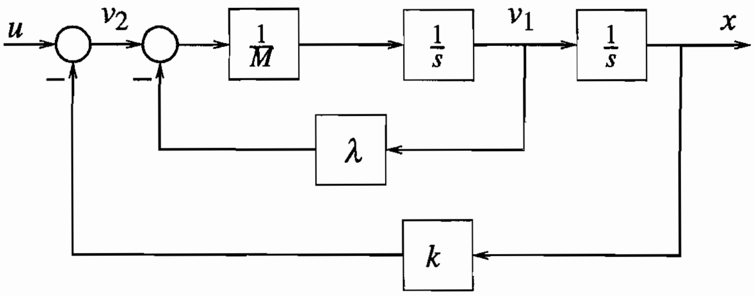

Fig. 2

(b) Assume $u = 0$. Suppose that the value of k is not known precisely and the uncertainty is modelled as $k + \delta_1$ such that $k + \delta_1 > 0$.

(i) Show that the system is stable for any δ_1 such that $k + \delta_1 > 0$. [10%]

(ii) Consider Fig. 3 and find the transfer function from w_1 to z_1 (assuming $w_2 = 0$). By noticing that $w_1 = \delta_1 z_1$, use the small gain theorem to find a condition on δ_1 for the system to be stable. Explain why this result and the one obtained in Part (b)(i) are different. [30%]

(iii) Assume that in addition there is an uncertainty in λ which is modelled as $\lambda + \delta_2$ such that $\lambda + \delta_2 > 0$, and consider the map $w_2 = \delta_2 z_2$. Show that

(cont.

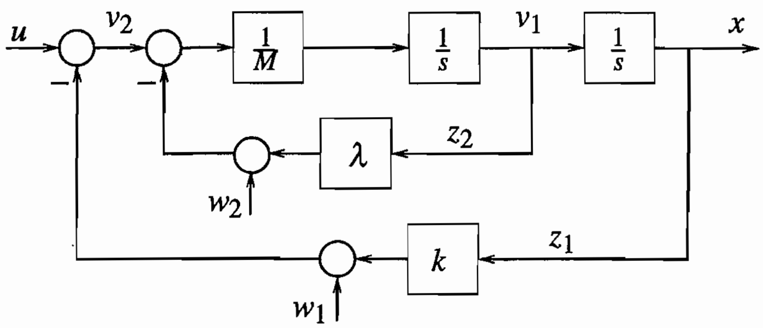

Fig. 3

the transfer function $G(s)$ from $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ is given by

$$G(s) = \frac{-1}{s^2M + s\lambda + k} \begin{bmatrix} 1 \\ s \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$$

[20%]

(iv) Find an upper bound of $\mu(G(j\omega))$ by computing $\sup_\omega \overline{\sigma}(G(j\omega))$, where μ is the structured singular value. [20%]

(v) Find $\max_\omega \mu(G(j\omega))$ given the uncertainty structure $\Delta = \mathrm{diag}(\delta_1, \delta_2)$ with δ_1 and δ_2 such that $k + \delta_1 > 0$ and $\lambda + \delta_2 > 0$. [10%]

2 Consider a generalized plant $P(s)$ with realization

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = Ax + B_1 w + B_2 u$$

$$z = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = \begin{bmatrix} C_1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = x$$

(a) Solve the CARE given by

$$XA + A^T X + C_1^T C_1 - XB_2 B_2^T X = 0$$

for this system and find its symmetric, real, stabilizing solution. [20%]

(b) Find the stabilizing optimal controller for the problem:

$$\min_{K(s)\text{ stabilizing}} \|\mathscr{F}_l(P(s), K(s))\|_2$$

where $\mathscr{F}_l(\cdot,\cdot)$ denotes the lower linear fractional transformation, and compute the corresponding optimal value of $\|\mathscr{F}_l(P(s), K(s))\|_2$. [20%]

(c) For the generalized plant $P(s)$ given above, the loop from y to u is closed with a static feedback controller $K(s)$ given by $u = \begin{bmatrix} k_1 & k_2 \end{bmatrix} y$.

(i) Show that the corresponding $\mathscr{F}_l(P(s), K(s))$ is stable for all $k_1 < 0$ and $k_2 < 0$. [10%]

(ii) Show that, for the closed loop system, the observability gramian from w to z is given by

$$L = \begin{bmatrix} \frac{-k_1(k_1^2+4)+4k_2^2}{2k_1 k_2} & -\frac{k_1^2+4}{2k_1} \\ -\frac{k_1^2+4}{2k_1} & \frac{(k_1^2+4)-k_1 k_2^2}{2k_1 k_2} \end{bmatrix}.$$

(Hint: The observability gramian L satisfies the equation

$$\tilde{A}^T L + L\tilde{A} + \tilde{C}^T \tilde{C} = 0$$

where $\tilde{A}$ and $\tilde{C}$ correspond to the closed-loop system state-space realization.) [20%]

(cont.

(iii) Compute the $\mathscr{H}_2$ norm of the transfer function $\mathscr{F}_l(P(s), K(s))$ corresponding to $u = \begin{bmatrix} k_1 & k_2 \end{bmatrix} y$. [20%]

(iv) Give the values of k_1 and k_2 corresponding to

$$\min_{K(s) \text{ stabilizing}} \|\mathscr{F}_l(P(s), K(s))\|_2 .$$

(Hint: look at Part (b).) [10%]

(TURN OVER

3 Consider the finite horizon optimal control problem

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$$

$$J(x_0, u(\cdot)) = \int_0^T c(x(t), u(t))dt + J_T(x(T))$$

where the control function $u(\cdot) : [0, T] \to \mathbb{R}^m$ is to be chosen to minimize J.

(a) Define the "value function" $V(x,t)$ for this problem and explain the significance of $V(x,T)$ and $V(x,0)$. [20%]

(b) Give the dynamic programming equation that $V(x,t)$ must satisfy, describe the principle that this equation captures, and explain how it simplifies the problem. [20%]

(c) Consider now the particular optimal control problem:

$$\dot{x}(t) = x(t) + u(t), \quad x(0) = x_0 \quad (n = 1, m = 1)$$

$$J(x_0, u(\cdot)) = \int_0^T (u(t))^2 dt + (x(T))^2$$

(i) State the dynamic programming equation for this problem and show that it belongs to the class of Bernoulli differential equations $\dot{\alpha}(t) = c\alpha(t) + d\alpha^k(t)$ where c and d are constants and k is an integer. (Hint: you may need to solve the following equation

$$-\dot{X} = Q + XA + A^T X - XBR^{-1}B^T X$$

with the terminal condition $X(T) = 1$.) [30%]

(ii) Find the optimal cost and the optimal control for $T = 10$ and $x_0 = 2$. (Hint: the change of variable $\beta(t) = \alpha^{1-k}(t)$ may be useful to solve the Bernoulli differential equation) [30%]

END OF PAPER