ENGINEERING TRIPOS PART IIB

Monday 28 April 2008 9 to 10.30

Module 4F7

DIGITAL FILTERS AND SPECTRUM ESTIMATION

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) Describe the Recursive Least Square (RLS) method for adaptive filtering. Your answer should include the following points: a definition of the relevant signals and the cost function being minimised; the influence of the forgetting factor on tracking performance and misadjustment; whether the RLS method converges to the Wiener filter for a particular value of forgetting factor and why. [35%]

(b) Figure 1 depicts an impulse response identification problem. The coefficients of the FIR filter are to be identified using a known input sequence $\{u(n)\}_{n\geq 0}$. The output of the filter $y(n)$ is measured by an imperfect sensor, which can be modelled as the filter output with additive noise $v(n)$. Assume that $E\{v(n)\} = 0$, $E\{v(n)^2\} = \sigma_v^2$ and that the impulse response of the filter to be identified is $[\beta_0, \ldots, \beta_{L-1}]^{\mathrm{T}}$.

Explain how to solve the identification problem using RLS, assuming knowledge of L. For the case $L = 2$ and given that $\beta_0 = 1$, compute the RLS solution at time n. [35%]

(c) Explain how to solve for $[\beta_0, \ldots, \beta_{L-1}]^{\mathrm{T}}$ using the Steepest Descent and the Least Mean Square (LMS) algorithms. Assuming that $u(n) = \sum_{i=0}^{M-1} \alpha_i w(n-i)$ where $E\{w(n)\} = 0$, $E\{w(n)^2\} = \sigma_w^2$, $E\{w(k)w(l)\} = 0$ when $l \neq k$, what is the stability condition for the LMS algorithm? (Hint: use an appropriate estimate of $\lambda_{\max}$.) [30%]

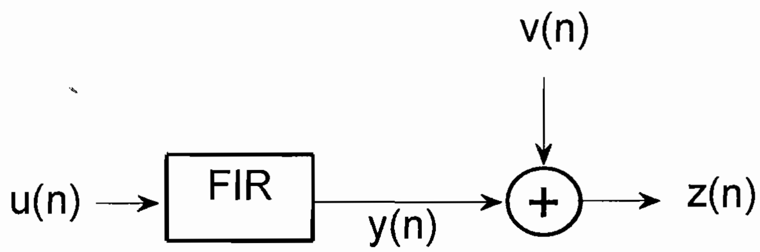

Fig. 1

2 (a) Consider an infinite collection $\{y_t\}_{t\geq 0}$ of random variables where

$$y_t = x + v_t.$$

Describe the Gram-Schmidt orthogonalization procedure for forming the minimum variance linear estimate of x using the first T random variables from this collection.

For the case $T = 2$, you are given $E(xy_1) = 0.1$, $E(xy_2) = 0.3$, $E(y_1 y_2) = -0.2$, $E(y_1^2) = 1$, $E(y_2^2) = 1.5$. Compute the coefficient of y_1 and y_2 of the estimator given by the Gram-Schmidt procedure. (Hint: write $\hat{x}$ as a function of y_1 and y_2 explicitly and compute the coefficients.) [35%]

(b) A constant temperature θ is measured with the use of noisy sensors. The measurement made by sensor i is $y_i = \theta + v_i$ where v_i is a Gaussian random variable with mean 0 and variance σ_i^2. Assume random variables $\{v_i\}_{i=1,2,\ldots,T}$ are independent. Derive the minimum variance linear unbiased estimator for $T = 2$ and compute the minimum variance. [35%]

(c) Let the i th sensor variance be $\sigma_i^2 = i$. Using the answer derived in the previous part, determine the value of T that will reduce the variance of the estimator to less than 6/11. [30%]

3 (a) Describe the *parametric* approach to power spectrum estimation, including a brief discussion of the ARMA model, the AR model, and their power spectra. [30%]

(b) Write down an expression for the prediction error at time index n for an autoregressive model of order P. [5%]

By considering minimisation of an appropriate function of this prediction error corresponding to a finite length of data $x_0, \ldots, x_{N-1}$, show that the vector of autoregressive parameters $\mathbf{a}$ may be estimated as

$$\mathbf{a} = -(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{x}$$

where $\mathbf{x}$ and $\mathbf{X}$, which are a vector and matrix containing observed data values, should be carefully defined. [15%]

Explain how the covariance and autocorrelation methods can be obtained from this method and briefly summarise the properties of each. [10%]

(c) A discrete time function is defined as:

$$x_0 = 1, x_1 = -0.9, x_2 = 0.81, \ldots$$

i.e. the general term is $x_n = (-0.9)^n$. Write down an autoregressive model with order $P = 1$ which fits this function perfectly (i.e. with zero prediction error for any $n > 0$). [10%]

Now compute estimates of order $P = 1$ autoregressive models from data points $x_0, x_1, \ldots x_{N-1}$, using both the autocorrelation method and the covariance method. [20%]

Comment on the similarity between these two estimates and also with the model obtained above which fits the data perfectly. [10%]

4 (a) Define the bias and variance of an estimator for a random quantity, explaining how they can be used to evaluate the estimator's performance. [20%]

(b) In a power spectrum estimation method for a wide-sense stationary and ergodic random process, the data is first windowed using a window function w_n having length N, i.e. $w_n = 0$ for $n < 0$ and $n > N-1$.

The autocorrelation function is then estimated as

$$\hat{R}_{XX}[k] = \frac{1}{N}\sum_{n=0}^{N-1-k}(w_n x_n)(w_{n+k}x_{n+k}), \text{ for } k = 0, 1, \dots, N-1,$$

$$\hat{R}_{XX}[k] = \hat{R}_{XX}[-k], \text{ for } k = -1, -2, \dots, -N+1$$

and $\hat{R}_{XX}[k] = 0$ for all values of $|k| > N-1$, where the x_n are measured values drawn from the random process.

(i) Show that the expected value of the autocorrelation function estimate is given by

$$E[\hat{R}_{XX}[k]] = R_{XX}[k]\frac{1}{N}\sum_{n=0}^{N-1-k}(w_n w_{n+k})$$

where $R_{XX}[k]$ is the true autocorrelation function for the process.

Is this an unbiased estimate? [20%]

(ii) The power spectrum estimate $\hat{S}_X(e^{j\omega T})$ is then obtained by taking the DTFT of the estimated autocorrelation function $\hat{R}_{XX}[k]$.

Show that the expected value of the corresponding power spectrum estimate is:

$$E[\hat{S}_X(e^{j\omega T})] = \frac{1}{2\pi N}S_X(e^{j\omega T}) * |W(e^{j\omega T})|^2$$

where $S_X(e^{j\omega T})$ is the true power spectrum of the random process, $W(e^{j\omega T})$ is the DTFT of the window function w_n, and $*$ denotes the convolution operator. [40%]

(iii) How might this modified autocorrelation method be used to improve the performance and properties of the standard periodogram estimator? [20%]

END OF PAPER