ENGINEERING TRIPOS PART IIA
ENGINEERING TRIPOS PART IIB

Friday 2 May 2008 2.30 to 4

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: Data Sheet for 4M12 (3 sides).

STATIONERY
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

(TURN OVER

1 (a) Use suffix notation to find alternative expressions for

(i) $\mathbf{a}\times(\mathbf{b}\times(\mathbf{a}\times\mathbf{c}))$ [20%]

(ii) $(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c})$ [20%]

where **a**, **b** and **c** are constant vectors.

Discuss the circumstances under which these two quantities can be equal. [10%]

(b) A closed surface S_1 lies inside another closed surface S_2. Explain carefully how the divergence theorem can be applied to the volume V lying between these two surfaces. [15%]

Hence show that the integral

$$\iint(\nabla\times\mathbf{f})\cdot d\mathbf{A}$$

has the same value when taken over surface S_1 or surface S_2, where **f** is any vector function of position. [35%]

2 (a) The integral

$$I = \int_0^a F(y, y', x)\,dx$$

where $\dfrac{dy}{dx}$ is denoted y', is to be minimised by choosing the appropriate function $y(x)$. Explain briefly how this leads to a differential equation which $y(x)$ must satisfy, and find the form of the boundary condition which must be satisfied at $x = a$ if the value of y is not constrained there. [30%]

(b) A frictionless wire lies in a vertical plane which connects two points A and B, A being higher than B. The point A is fixed at the point $x = 0$, $y = 0$, while the point B can lie anywhere on the vertical line $x = a$. A particle of mass m slides down the wire under the effect of gravity, starting from rest at the point A. The shape of the wire has the form $y(x)$, where x is the usual horizontal Cartesian coordinate and y is measured vertically downwards. The shape $y(x)$ is to be found which minimises the time taken for the particle to reach B.

(i) Show that the function $y(x)$ must minimise the integral

$$I = \int_0^a \sqrt{\frac{1 + y'^2}{y}}\,dx,$$

and find the boundary condition which must be satisfied at $x = a$. [30%]

(ii) Using a first integral of the Euler-Lagrange equation, show that the curve $y(x)$ must have the parametric form

$$x = b(2\theta - \sin 2\theta) + x_0, \qquad y = b(1 - \cos 2\theta)$$

where b and x_0 are constants. Determine the values of these constants, and hence find the distance $y(a)$. [40%]

3 (a) Explain how equations of the form

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$$

can be solved by integrating along characteristic lines. [20%]

(b) The function $u(x,y)$ satisfies the equation

$$\frac{\partial u}{\partial x} + xe^{-y}\frac{\partial u}{\partial y} = u \quad \text{for } x \geq 0 \text{ and for all } y.$$

If $u(0,y) = e^y$, show, by integrating along characteristic lines, that u is given by

$$u = e^{x+y} - \frac{x^2}{2}e^x \ .$$ [40%]

(c) Explain carefully, with the aid of a sketch, for which part of the domain $x \geq 0$ this solution is valid. [20%]

(d) The domain is reduced to the quadrant $x \geq 0$ and $y \geq 0$. Determine $u(x,0)$ to make the solution

$$u = e^{x+y} - \frac{x^2}{2}e^x$$

for the same region as part (c), and $u = 0$ elsewhere within the quadrant. [20%]

4 (a) Derive the general representation theorem

$$u(\underline{x}_0) = \int_V G(\underline{x},\underline{x}_0) f(\underline{x}) dV + \int_S u(\underline{x}) \frac{\partial G(\underline{x},\underline{x}_0)}{\partial n} dS - \int_S G(\underline{x},\underline{x}_0) \frac{\partial u(\underline{x})}{\partial n} dS$$

where $u(\underline{x})$ is a solution of the Poisson Equation

$$\nabla^2 u = f(\underline{x})$$

in a region V, which is surrounded by the surface S, and where $G(\underline{x},\underline{x}_0)$ is a Green Function. Explain how this representation theorem is used to derive solutions for u which satisfy the boundary condition $u = U(\underline{x})$ for $\underline{x}$ on the boundary S. [25%]

(b) Show that the Green Function for the half-space problem

$$\nabla^2 u = 0 \qquad \text{for } z \geq 0 \text{ for all } x \text{ and } y,$$

with $u(x,y,0) = U(x,y)$, can be written

$$G(\underline{x},\underline{x}_0) = -\frac{1}{4\pi|\underline{x}-\underline{x}_0|} + \frac{1}{4\pi|\underline{x}-\underline{x}_1|}$$

for a suitable value of $\underline{x}_1$. [15%]

(c) Hence show that

$$u(\underline{x}_0) = \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \frac{z_0}{2\pi} \frac{U(x,y)}{\left[(x-x_0)^2 + (y-y_0)^2 + z_0^2\right]^{3/2}} dxdy \qquad [40\%]$$

(d) Explain why u is much smaller in magnitude, at large distances from the plane $z = 0$, than v, where v is the solution to

$$\nabla^2 v = 0 \qquad \text{for } z \geq 0 \text{ for all } x \text{ and } y,$$

with $\dfrac{\partial v(x,y,0)}{\partial z} = U(x,y)$. [20%]

END OF PAPER

Engineering Tripos Part IIA/Part IIB
Module 4M12: Partial differential equations and variational methods

DATA SHEET

Suffix notation

1 $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

2 $\varepsilon_{ijk} = \begin{cases} 0 & \text{if any two of } i,j,k \text{ are equal} \\ 1 & \text{if } (ijk) \text{ is a permutation of } (123) \\ -1 & \text{if } (ijk) \text{ is a permutation of } (321) \end{cases}$

3 $[\mathbf{x} \times \mathbf{y}]_i = \varepsilon_{ijk} x_j y_k$

4 $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

5 $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}$

6 $[\text{grad}\ \phi]_i = \dfrac{\partial \phi}{\partial x_i}$, $\text{div}\ \mathbf{u} = \dfrac{\partial u_i}{\partial x_i}$, $[\text{curl}\ \mathbf{u}]_i = \varepsilon_{ijk}\dfrac{\partial u_k}{\partial x_j}$

where ϕ is any scalar field and $\mathbf{u}$ is any vector field.

Integral theorems

7 Divergence theorem in general form:

$$\iiint_V \frac{\partial}{\partial x_i}(\text{anything})\, dV = \iint_S (\text{anything})\, dA_i$$

where V is a volume enclosed by a surface S, and (anything) denotes any legal suffix notation expression. The integration element $d\mathbf{A}$ denotes the vector element of area: it can also be written $\mathbf{n}\, dA$, where $\mathbf{n}$ is the outward-pointing unit normal vector to the surface S, and dA is the scalar element of area on that surface.

8 Stokes's theorem in general form:

$$\iint_S \varepsilon_{ijk} \frac{\partial}{\partial x_j}(\text{anything})\, dA_i = \oint_C (\text{anything})\, dl_k$$

where S is a region of (possibly curved) surface with a curve C running around the boundary, and (anything) denotes any legal suffix notation expression.

Variational methods

9 To minimise $I=\int_0^L F(y,y',x)\,dx$ where F is any function of $y(x)$ and its derivative, and x, F must satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left[\frac{\partial F}{\partial y'}\right]=0$$

at all positions x.

10 First integrals of the Euler-Lagrange equation:
If F depends on y' but not on y, then $\frac{\partial F}{\partial y'}=\text{constant}$.

If F does not depend explicitly on x (i.e. only depends on x via y and y') then

$$F-y'\frac{\partial F}{\partial y'}=\text{constant}.$$

11 Rayleigh's principle : for a linear vibrating system with potential energy V and kinetic energy $\omega^2\tilde{T}$ for harmonic motion at frequency ω, the quotient $\frac{V}{\tilde{T}}$ is stationary with respect to small perturbations when the motion is a normal mode, and its value is then equal to the squared natural frequency of that mode.

Partial Differential Equations

12 Classification : The second order quasi-linear partial differential equation

$$a\frac{\partial^2 u}{\partial x^2}+2b\frac{\partial^2 u}{\partial x\partial y}+c\frac{\partial^2 u}{\partial y^2}+F\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0$$

is:

hyperbolic	where	$b^2-ac>0$
parabolic	where	$b^2-ac=0$
elliptic	where	$b^2-ac<0$

13 Well-posed problem: a problem is well-posed if the solution
(i) exists
(ii) is unique
(iii) depends continuously on the input data (i.e. is stable to changes in the input data)

14 Common reference equations are:

Helmoltz Equation	$\nabla^2 u + k^2 u = 0$
Poisson Equation	$\nabla^2 u = f(x,y)$
Laplace Equation	$\nabla^2 u = 0$
Wave Equation	$\frac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0$
Diffusion Equation	$\frac{\partial u}{\partial t} - \alpha \nabla^2 u = 0$

The form of the Laplacian operator ∇^2 in various co-ordinate systems can be found in the Maths Data Book.

15 D'Alembert travelling wave solution: the solution of

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} x \qquad \text{for } t > 0 \text{ and for all } x$$

with initial conditions $u(x,0) = \phi(x)$ and $\frac{\partial u}{\partial t}(x,0) = \psi(x)$

is $u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi) d\xi$

16 Fundamental solution (Free-space Green's function):

2-D Poisson/Laplace Equation

$$\nabla^2 G(\underline{x}, \underline{x}_0) = \delta(\underline{x} - \underline{x}_0) \qquad G(\underline{x}, \underline{x}_0) = \frac{1}{2\pi} \ln |\underline{x} - \underline{x}_0|$$

3-D Poisson/Laplace Equation

$$\nabla^2 G(\underline{x}, \underline{x}_0) = \delta(\underline{x} - \underline{x}_0) \qquad G(\underline{x}, \underline{x}_0) = -\frac{1}{4\pi |\underline{x} - \underline{x}_0|}$$

Fundamental solution:
Diffusion Equation

$$\frac{\partial F}{\partial t} - \alpha \frac{\partial^2 F}{\partial x^2} = \delta(x - x_0)\delta(t - t_0)$$

$$F(x,t;x_0,t_0) = \frac{1}{\sqrt{4\alpha\pi(t-t_0)}} \exp\left(-\frac{(x-x_0)^2}{4\alpha(t-t_0)}\right) \text{ for } t > t_0$$

3-(space) D Wave Equation

$$\frac{\partial^2 F}{\partial t^2} - c^2 \nabla^2 F = \delta(t - t_0)\delta(\underline{x} - \underline{x}_0)$$

$$F(\underline{x},t;\underline{x}_0,t_0) = \frac{\delta\left(t - t_0 - \frac{|\underline{x} - \underline{x}_0|}{c}\right)}{4\pi c^2 |\underline{x} - \underline{x}_0|} \text{ for } t > t_0$$