ENGINEERING TRIPOS PART IIB
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Friday 9 May 2008 9 to 10.30

Module 4M13

COMPLEX ANALYSIS AND OPTIMIZATION

*Answer not more than **three** questions.*

The questions may be taken from any section.

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
4M13 datasheet (4 pages).

Answers to Sections A and B should be tied together and handed in separately.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1 (a) Suppose that a function f is differentiable in a domain S, apart from a finite set of poles, and has no zeros or poles on dS, a boundary of S. A zero of f has order k at a point z_0 if, in addition to $f(z_0) = 0$, the first $k-1$ derivatives of f also vanish at z_0, but $f^{(k)}(z_0) \neq 0$. Show that

(i) If f has a zero of order k at z_0, then f'/f has a pole there with residue k. [30%]

(ii) If f has a pole of order m at z_0, then f'/f has a pole there with residue $-m$. [30%]

(b) Hence show that

$$\frac{1}{2\pi i}\int_{dS}\frac{f'(z)}{f(z)}dz = N - P$$

where N and P are respectively the number of zeros and poles of f inside S, each counted according to its order. [25%]

(c) Verify the above formula for the polynomial $4z^2 - 1$ where S is the unit circle. [15%]

2 Calculate the following integrals:

(a) $\int_{-\infty}^{\infty} \frac{x\sin(\pi x)}{(1-x^2)} dx$. [50%]

(b) $\int_{0}^{\pi} \frac{1}{(1+b\cos^2 x)} dx$ for $b > -1$. [50%]

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SECTION B

3 A body of water of breadth b is to be crossed by a multi-span suspension bridge. The designer hopes to determine the span length L and cable dip d, as shown in Fig. 1, that will give the minimum cost for the whole crossing of n spans, where $b = nL$. For preliminary design the total cost of the bridge is assumed to have only two variable components: the cost of the suspension cables and the cost of the towers. The cost of the cables is taken to be directly proportional to the volume of steel required. The cost of each tower is assumed to be dominated by the cost of its subsea foundations and therefore independent of both L and d. The bridge deck is assumed to be subject to continuous load w per unit length across the entire span. Relevant design and cost data for the bridge are given in Table 1.

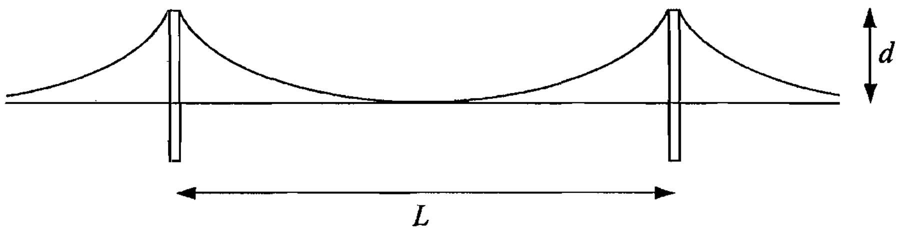

Fig. 1

(a) If the discrete number of spans n is treated as a continuous variable equal to b/L, show that a suitable objective function f for the cost of the bridge in terms of L and d is

$$f = \frac{bwc_{cable}}{8\sigma_d}\left(\frac{L^2}{d} + \frac{8}{3}d\right) + C_{tower}\left(\frac{b}{L} + 1\right).$$

You can assume that the length of cable used between the anchorages on the banks and the towers at either end of the bridge is negligible compared to that used in the spans and that the stress in the cable is equal to design stress σ_d. [15%]

(b) From a starting point of $L = 2000\,\text{m}$ and $d = 100\,\text{m}$ execute a single step of the steepest descent algorithm to find an improved design. [50%]

(c) By considering first and second-order optimality conditions find the minimum cost design. [20%]

(d) In the light of your answer to (c) comment on the performance of the steepest descent method and the realism of the cost model. [15%]

Crossing breadth	$b = 10{,}000\,\text{m}$
Bridge deck design load	$w = 160\,\text{kNm}^{-1}$
Design stress for high tensile steel cable	$\sigma_d = 1000\,\text{MPa}$
Length of cable in a single span	$s \approx L\left(1 + \dfrac{8d^2}{3L^2}\right)$
Tension in suspension cables	$T = \dfrac{wL^2}{8d}$
Unit cost of high tensile steel cable	$c_{cable} = £100{,}000\,\text{m}^{-3}$
Cost of each tower	$C_{tower} = £6.532$ million

Table 1

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4 An engineer has been asked to design a minimum weight, vertical, tubular, cantilever column of fixed length L to support a defined vertical load P without buckling or overstressing. The design of the column in shown schematically in Fig. 2. The inner radius R_i and outer radius R_o of the column can be varied to optimize the design.

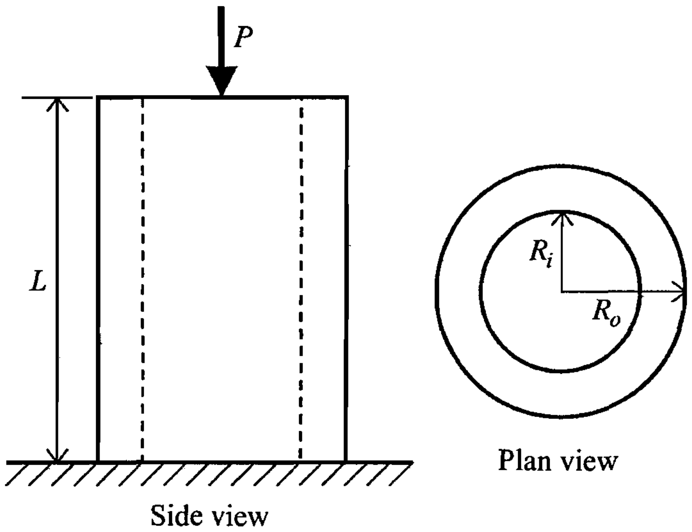

Fig. 2

The buckling load for such a column is $\frac{\pi^3 E}{16L^2}\left(R_o^{\,4} - R_i^{\,4}\right)$ and the maximum load it can withstand without overstressing is $\sigma_y \pi\left(R_o^{\,2} - R_i^{\,2}\right)$, where E and σ_y are respectively the Young's modulus and yield stress of the material from which the column is made. The effects of self-weight can be neglected.

(a) Formulate the task of optimizing the design of the column as a constrained minimization problem in standard form. [10%]

(b) For the case where $L = \pi\,\text{m}$, $P = 7.5 \times 10^{-3}\pi\sigma_y$ and $E = 1280\sigma_y$, identify the feasible region graphically. By superimposing contours of the objective function, identify which of the constraints are active at the optimum, and thus the optimal values of R_i and R_o for this design problem. [50%]

(cont.

(c) Confirm your results to (b) by considering the possible solutions obtained when using the Kuhn-Tucker multiplier method to solve the general case of this optimization problem, i.e. with general values of *L*, *P* and *E*. [40%]

END OF PAPER

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OPTIMIZATION

DATA SHEET

1. Taylor Series Expansion

For one variable:

$$f(x) = f(x^*) + (x - x^*) f'(x^*) + \frac{1}{2}(x - x^*)^2 f''(x^*) + R$$

For several variables:

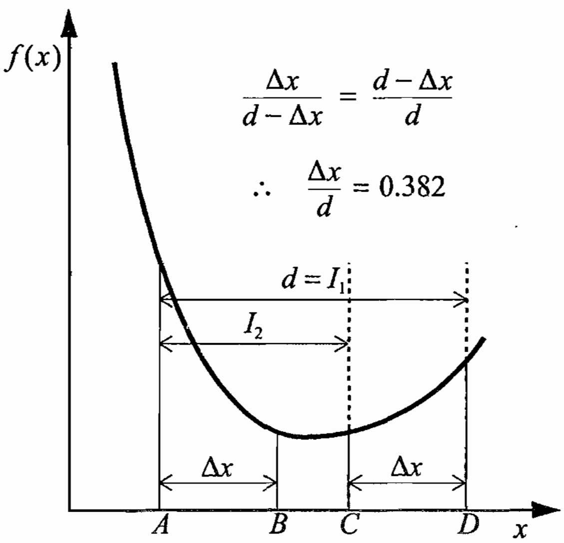
$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \boldsymbol{H}(\mathbf{x}^*) (\mathbf{x} - \mathbf{x}^*) + R$$

where

$$\text{gradient } \nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \quad \text{and hessian } \boldsymbol{H}(\mathbf{x}) = \nabla(\nabla f(\mathbf{x})) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$\boldsymbol{H}(\mathbf{x}^*)$ is a symmetric $n \times n$ matrix and R includes all higher order terms.

2. Golden Section Method



(a) Evaluate $f(x)$ at points A, B, C and D.

(b) If $f(B) < f(C)$, new interval is $A - C$.

If $f(B) > f(C)$, new interval is $B - D$.
If $f(B) = f(C)$, new interval is either $A - C$ or $B - D$.

(c) Evaluate $f(x)$ at new interior point. If not converged, go to (b).

3. **Newton's Method**

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -H(\mathbf{x}_k)^{-1}\nabla f(\mathbf{x}_k)$

(c) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k$

(d) Test for convergence. If not converged, go to step (b)

4. **Steepest Descent Method**

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -\nabla f(\mathbf{x}_k)$

(c) Perform line search to determine step size α_k or evaluate $\alpha_k = \dfrac{\mathbf{d}_k^T\mathbf{d}_k}{\mathbf{d}_k^T H(\mathbf{x}_k)\mathbf{d}_k}$

(d) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k\mathbf{d}_k$

(e) Test for convergence. If not converged, go to step (b)

5. **Conjugate Gradient Method**

(a) Select starting point $\mathbf{x}_0$ and compute $\mathbf{d}_0 = -\nabla f(\mathbf{x}_0)$ and $\alpha_0 = \dfrac{\mathbf{d}_0^T\mathbf{d}_0}{\mathbf{d}_0^T H(\mathbf{x}_0)\,\mathbf{d}_0}$

(b) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k\mathbf{d}_k$

(c) Evaluate $\nabla f(\mathbf{x}_{k+1})$ and $\beta_k = \left[\dfrac{|\nabla f(\mathbf{x}_{k+1})|}{|\nabla f(\mathbf{x}_k)|}\right]^2$

(d) Determine search direction $\mathbf{d}_{k+1} = -\nabla f(\mathbf{x}_{k+1}) + \beta_k\mathbf{d}_k$

(e) Determine step size $\alpha_{k+1} = -\dfrac{\mathbf{d}_{k+1}^T\nabla f(\mathbf{x}_{k+1})}{\mathbf{d}_{k+1}^T H(\mathbf{x}_{k+1})\,\mathbf{d}_{k+1}}$

(f) Test for convergence. If not converged, go to step (b)

6. **Gauss-Newton Method (for Nonlinear Least Squares)**

If the minimum squared error of residuals $\mathbf{r}(\mathbf{x})$ is sought:

$$\text{Minimise } f(\mathbf{x}) = \sum_{i=1}^{m} r_i^2(\mathbf{x}) = \mathbf{r}(\mathbf{x})^T\mathbf{r}(\mathbf{x})$$

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -[\,J(\mathbf{x}_k)^T J(\mathbf{x}_k)\,]^{-1}\,J(\mathbf{x}_k)^T\mathbf{r}(\mathbf{x}_k)$

$$\text{where } J(\mathbf{x}) = \begin{bmatrix} \nabla r_1(\mathbf{x})^T \\ \vdots \\ \nabla r_m(\mathbf{x})^T \end{bmatrix} = \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_m}{\partial x_1} & \cdots & \frac{\partial r_m}{\partial x_n} \end{bmatrix}$$

(c) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k$

(d) Test for convergence. If not converged, go to step (b)

7. **Lagrange Multipliers**

To minimise $f(\mathbf{x})$ subject to m equality constraints $h_i(\mathbf{x}) = 0, i = 1, \ldots, m$, solve the system of simultaneous equations

$$\begin{aligned} \nabla f(\mathbf{x}^*) + [\nabla \mathbf{h}(\mathbf{x}^*)]^T \boldsymbol{\lambda} &= 0 \quad (n \text{ equations}) \\ \mathbf{h}(\mathbf{x}^*) &= 0 \quad (m \text{ equations}) \end{aligned}$$

where $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_m]^T$ is the vector of Lagrange multipliers and

$$[\nabla \mathbf{h}(\mathbf{x}^*)]^T = \begin{bmatrix} \nabla h_1(\mathbf{x}^*) & \ldots & \nabla h_m(\mathbf{x}^*) \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_1}{\partial x_n} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$$

8. **Kuhn-Tucker Multipliers**

To minimise $f(\mathbf{x})$ subject to m equality constraints $h_i(\mathbf{x}) = 0, i = 1, \ldots, m$ and p inequality constraints $g_i(\mathbf{x}) \le 0, i = 1, \ldots, p$, solve the system of simultaneous equations

$$\begin{aligned} \nabla f(\mathbf{x}^*) + [\nabla \mathbf{h}(\mathbf{x}^*)]^T \boldsymbol{\lambda} + [\nabla \mathbf{g}(\mathbf{x}^*)]^T \boldsymbol{\mu} &= 0 \quad (n \text{ equations}) \\ \mathbf{h}(\mathbf{x}^*) &= 0 \quad (m \text{ equations}) \\ \forall\ i = 1, \ldots, p, \quad \mu_i g_i(\mathbf{x}) &= 0 \quad (p \text{ equations}) \end{aligned}$$

where $\boldsymbol{\lambda}$ are Lagrange multipliers and $\boldsymbol{\mu} \ge 0$ are the Kuhn-Tucker multipliers.

9. **Penalty & Barrier Functions**

To minimise $f(\mathbf{x})$ subject to p inequality constraints $g_i(\mathbf{x}) \le 0,\ i = 1, \ldots, p$, define

$$q(\mathbf{x}, p_k) = f(\mathbf{x}) + p_k P(\mathbf{x})$$

where $P(\mathbf{x})$ is a penalty function, e.g.

$$P(\mathbf{x}) = \sum_{i=1}^{p} (\max[0,\ g_i(\mathbf{x})])^2$$

or alternatively

$$q(\mathbf{x}, p_k) = f(\mathbf{x}) - \frac{1}{p_k} B(\mathbf{x})$$

where $B(\mathbf{x})$ is a barrier function, e.g.

$$B(\mathbf{x}) = \sum_{i=1}^{p} \frac{1}{g_i(\mathbf{x})}$$

Then for successive $k = 1, 2, \ldots$ and p_k such that $p_k > 0$ and $p_{k+1} > p_k$, solve the problem

minimise $q(\mathbf{x}, p_k)$