2009 PART IIB 4A8 ENVIRONMENTAL FLUID MECHANICS

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1

Geostrophic flow equation

$$2\underline{\Omega} \times \underline{u} = -\frac{1}{\rho_0} \nabla P \quad \text{——} \quad (1)$$

(a) The pressure gradient force is balanced by the Coriolis force in a rotating frame of reference.

No, this equation needs to be modified by including viscous forces, ~~are~~ which are important inside the boundary layers. The modified equation is given in the data card as Ekman layer flow equation.

(b) $\underline{\omega} = \nabla \times \underline{u} \Rightarrow \omega_k = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

(1) in component form

$$2\Omega_z v = \frac{1}{\rho_0}\frac{\partial P}{\partial x} \quad \text{and} \quad 2\Omega_z u = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}$$

Taking appropriate derivatives of these two Components

$$\omega_k = \frac{1}{2\Omega_z \rho_0}\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2}\right) = \frac{1}{2\Omega_z \rho_0}\nabla^2 P.$$

Thus, ω_k is related to $\nabla^2 P$, the Pressure Laplacian.

$$\nabla \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

$$= \frac{1}{2\Omega_z \rho_0}\left(-\frac{\partial^2 p}{\partial x \partial y} + \frac{\partial^2 p}{\partial x \partial y}\right) = 0$$

$\because$ p is continuous

(c)

$$\frac{\partial p}{\partial z} = -\rho g = -\rho_0 (1 - \alpha \Theta) g$$

Taking z-derivative of the components of (1)

$$2\Omega_z \frac{\partial v}{\partial z} = \frac{1}{\rho_0}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right) = g\alpha \frac{\partial \Theta}{\partial x}$$

$$2\Omega_z \frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial z}\right) = -g\alpha \frac{\partial \Theta}{\partial y}$$

Thus, the required equations for the thermal wind are

$$\boxed{2\Omega_z \frac{\partial v}{\partial z} = g\alpha \frac{\partial \Theta}{\partial x} \quad \& \quad 2\Omega_z \frac{\partial u}{\partial z} = -g\alpha \frac{\partial \Theta}{\partial y}}$$

The vertical shear of horizontal velocities ~~are~~ is along the isotherms, and it's magnitude is given by the horizontal gradient of the temperature increment Θ. $(\equiv (T - T_0))$

$$\theta = Ax^2 + By$$

$$\Rightarrow \frac{\partial \theta}{\partial x} = 2Ax \ ; \qquad \frac{\partial \theta}{\partial y} = B$$

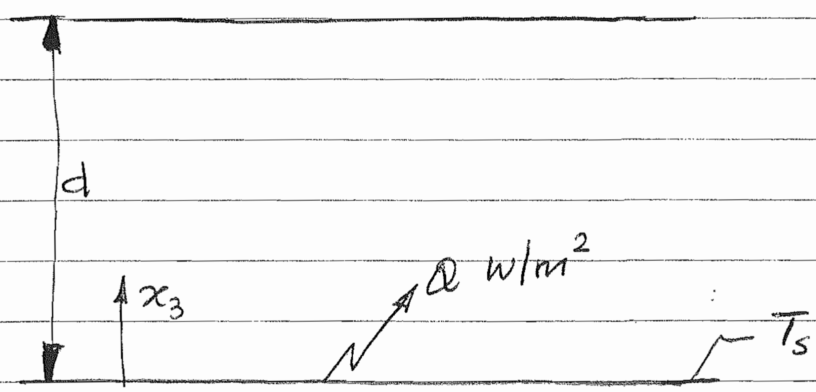
$$\therefore \frac{\partial v}{\partial z} = \frac{g\alpha}{\Omega_z} A x \quad \Rightarrow \quad \boxed{v = \frac{g\alpha A}{\Omega_z} x z + C_1(x,y)}$$

$$\frac{\partial u}{\partial z} = -\frac{g\alpha}{2\Omega_z} B \quad \Rightarrow \quad \boxed{u = \frac{g\alpha B}{2\Omega_z} z + C_2(x,y)}$$

To determine $C_1(x,y)$ & $C_2(x,y)$ ~~to~~ one needs to know u & v at some height z_1. Although u & v need to satisfy no slip condition at $z=0$, C_1 & C_2 can not be zero since viscous effects are not included in the analysis.

The velocities u & v vary linearly with vertical distance z.

2



a) Mean temperature equation

$$\rho C_p \frac{D\overline{T}}{Dt} = \nabla \cdot \left(\lambda \nabla \overline{T} - \overline{u\,\theta}\, \rho C_p \right)$$

Stationary flow
no mean motion (Velocities) $\Bigg\} \Rightarrow \quad \dfrac{D\overline{T}}{Dt} = 0.$

$$\Rightarrow \lambda \frac{d\overline{T}}{dx_3} - \overline{u_3\theta}\, \rho C_p = -Q$$

Neglect molecular heat flux

$$\Rightarrow \boxed{\overline{u_3\theta} = \frac{Q}{\rho C_p}}$$

b)

Turbulent Kinetic energy equation

$$\rho \frac{Dk}{Dt} = -\rho \overline{u_i u_k} \frac{\partial \overline{U}_i}{\partial x_k} - \rho \epsilon + \overline{f_i u_i}$$

No mean velocities & Stationary

$$\Rightarrow \boxed{\epsilon = \frac{\overline{f_3 u_3}}{\rho}} = \frac{g}{T} \overline{u_3 \theta}$$

dissipation balances Buoyancy

<u>Variance equation:</u>

$$\frac{D\sigma}{Dt} = -2\overline{u_j \theta} \frac{\partial \overline{T}}{\partial x_j} - \epsilon_\sigma + \text{molecular diffusion}$$

$$\Rightarrow \boxed{\epsilon_\sigma = -2\overline{u_3 \theta} \frac{\partial \overline{T}}{\partial x_3}}$$

dissipation balances production

$$\epsilon = \frac{g}{T} \overline{u_3 \theta} \Rightarrow \frac{u^3}{\lambda} \sim \frac{g}{T} \overline{u_3 \theta}$$

$$\Rightarrow u^2 \sim \left(\frac{g}{T} \lambda \overline{u_3 \theta}\right)^{2/3}$$

$$\boxed{k \sim \left(\frac{g}{T} d \overline{u_3 \theta}\right)^{2/3} \text{ or } \left(\frac{g}{T} \frac{d}{\rho} \frac{Q}{c_p}\right)^{2/3}}$$

$$\boxed{\epsilon \sim \frac{g}{T} \frac{Q}{\rho c_p}}$$

c)

$$\epsilon_\sigma \approx -2\overline{u_3\theta}\,\frac{d\overline{T}}{dx_3}$$

$$C_d\,\sigma\,\frac{\epsilon}{k} \approx -2\,\frac{Q}{\rho C_p}\,\frac{d\overline{T}}{dx_3}$$

from (b) $\quad \epsilon/k \approx \frac{1}{d}\left(\frac{g}{\overline{T}}\,\frac{d\,Q}{\rho C_p}\right)^{1/3}$

$$\Rightarrow C_d\,\sigma\,\frac{1}{d}\left(\frac{g\,d\,Q}{\rho C_p}\right)^{1/3}\overline{T}^{-1/3} = -2\left(\frac{Q}{\rho C_p}\right)\frac{d\overline{T}}{dx_3}$$

$$-A\,dx_3 = \overline{T}^{1/3}\,d\overline{T}$$

where $\quad A \equiv \frac{C_d}{2}\,\sigma\left(\frac{g}{d^2}\right)^{1/3}\left(\frac{\rho C_p}{Q}\right)^{2/3}$

integrate

$$\frac{3}{4}\,\overline{T}^{4/3}\Big|_{T_s}^{\overline{T}} = -A\,x_3$$

$$\overline{T}^{4/3} - T_s^{4/3} = -\frac{4}{3}\,A\,x_3$$

$$\frac{\overline{T}}{T_s} = \left(1 - A^* x_3\right)^{3/4}. \qquad A^* \equiv \frac{4}{3}\,\frac{A}{T_s^{4/3}}$$

@ $x_3 = d/2$

$$\boxed{\frac{\overline{T}}{T_s} = \left(1 - \mathcal{A}\,d\right)^{3/4}}$$

$$\mathcal{A} = A^*/2 = \frac{C_d}{3}\,\frac{\sigma}{T_s^{4/3}}\left(\frac{g}{d^2}\right)^{1/3}\left(\frac{\rho C_p}{Q}\right)^{2/3}$$

Q3

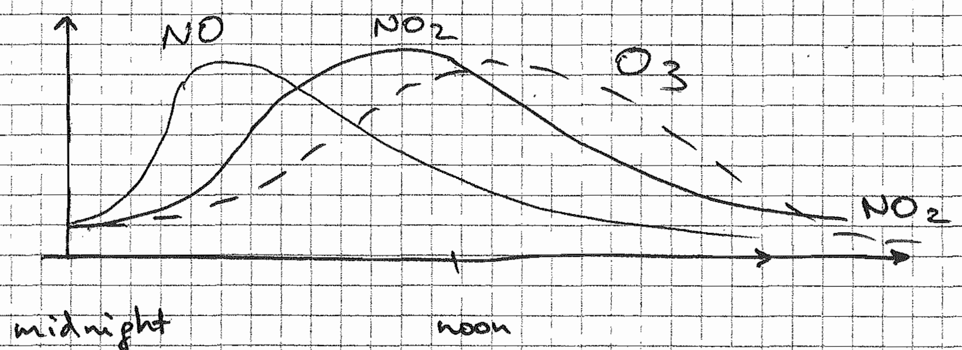
(a) NO: emitted from transport & power generation

O_3: not emitted, exists in background & created by photochemistry

NO_2: emitted little, created by $NO + O_3 \rightarrow NO_2 + O$

destroyed by photolysis $NO_2 \rightarrow NO + O$

$O + O_2 \rightarrow O_3$, which further reacts with NO.



O_3 is irritating for respiratory system and causes impaired vision.

NO_2 (brown gas) causes lung irritation at high concentrations and contibutes to acid rain

NO could cause liver damage in high doses

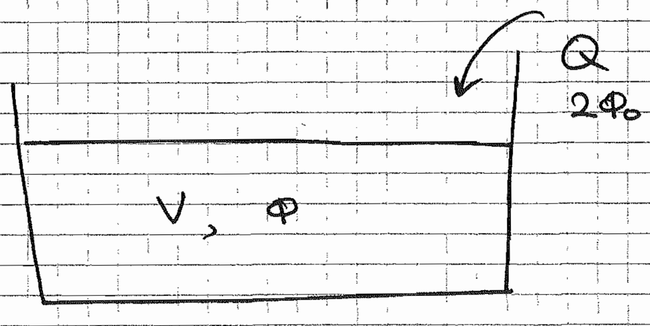
VOC: emitted by transport, industry, domestic sector.

participates in smog formation

$NO + VOC \rightarrow NO_2 + VOC^*$

Can be toxic, carcinogenic, irritant.

(b)



– Water conservation: $\dfrac{dV}{dt} = Q \Rightarrow V = V_0 + Qt$ (1)

– Pollutant conservation: $\dfrac{d(\Phi V)}{dt} = Q \cdot 2\Phi_0$

$$\Rightarrow \Phi \frac{dV}{dt} + V \frac{d\Phi}{dt} = Q\,\Phi_1 \qquad (\Phi_1 = 2\Phi_0)$$

$$\Leftrightarrow (V_0 + Qt) \frac{d\Phi}{dt} = Q\Phi_1 - \Phi \cdot Q$$

$$\Leftrightarrow \frac{d\Phi}{\Phi_1 - \Phi} = \frac{dt}{\frac{V_0}{Q} + t}$$

$$\Leftrightarrow -\ln(\Phi_1 - \Phi) = \ln\left(\frac{V_0}{Q} + t\right) + C$$

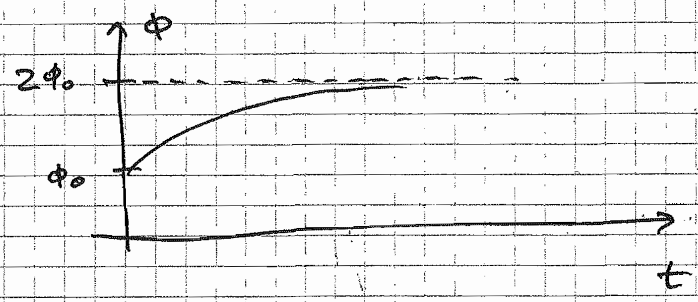
$$\Leftrightarrow \Phi_1 - \Phi = A\left(\frac{V_0}{Q} + t\right)^{-1}$$

At $t = 0$, $\Phi = \Phi_0$

& using $\Phi_1 = 2\Phi_0$, $\Phi_0 = A\dfrac{Q}{V_0} \Leftrightarrow A = \Phi_0 \dfrac{V_0}{Q}$

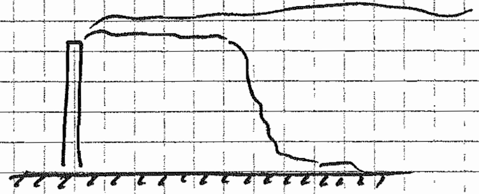
$$\therefore\ 2\Phi_0 - \Phi = \frac{\Phi_0 V_0}{Q}\left(\frac{V_0}{Q} + t\right)^{-1}$$

$$\Leftrightarrow \frac{\Phi}{\Phi_0} = 2 - \left(1 + \frac{tQ}{V_0}\right)^{-1} \qquad \text{Q.E.D.}$$

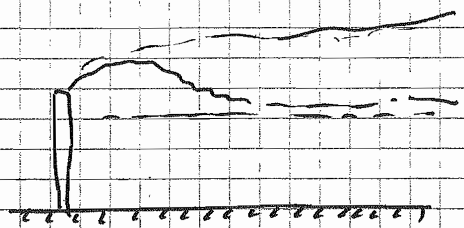


Q4

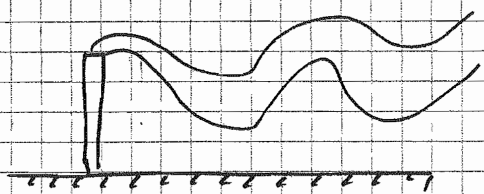
(a)



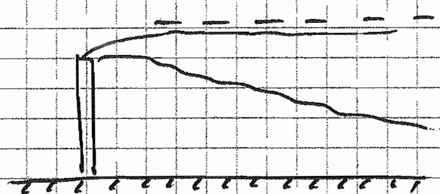
Fumigation: when mixing height increases at dawn and reaches undiluted plume



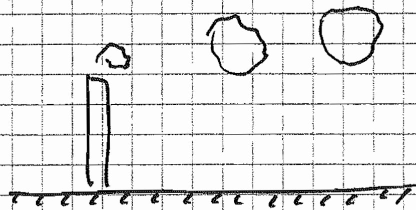
Lofting: source above inversion, plume avoids ground



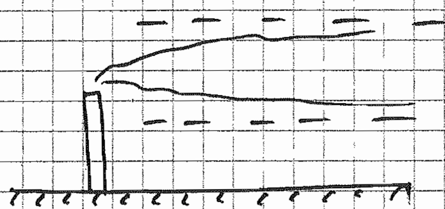
Looping: large-scale motion, vigorous turbulence neutral or unstable



Trapped plume: source below inversion, spreads to the ground



Thermalling: very buoyant plume, breaks apart



Fanning: plume between two inversions

(b) $$U \frac{\partial \Phi}{\partial x} = K \frac{\partial^2 \Phi}{\partial y^2} \qquad (1)$$

If $$\Phi = \frac{Q}{U \sigma \sqrt{2\pi}} \exp\left(-\frac{y^2}{2\sigma^2}\right)$$ Q: kg/s/m of source

then: $$\frac{\partial \Phi}{\partial y} = -2y\,\Phi\,\frac{1}{2\sigma^2} = -\frac{y}{\sigma^2}\Phi \Rightarrow \frac{\partial^2 \Phi}{\partial y^2} = -\frac{\Phi}{\sigma^2}\left(1 - \frac{y^2}{\sigma^2}\right)$$

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi}{\partial \sigma}\frac{d\sigma}{dx}$$

$$= \left[-\frac{\Phi}{\sigma} + \Phi\left(\frac{y^2}{\sigma^3}\right)\right]\frac{d\sigma}{dx} = -\frac{\Phi}{\sigma}\left(1 - \frac{y^2}{\sigma^2}\right)\frac{d\sigma}{dx}$$

Q4 cont'd

$$U \frac{\partial \Phi}{\partial x} = K \frac{\partial^2 \Phi}{\partial y^2} \Leftrightarrow U\left(-\frac{\Phi}{\sigma}\right)\left(1-\frac{y^2}{\sigma^2}\right)\frac{d\sigma}{dx} = K\left(-\frac{\Phi}{\sigma^2}\right)\left(1-\frac{y^2}{\sigma^2}\right)$$

$$\Leftrightarrow \quad \sigma \frac{d\sigma}{dx} = \frac{K}{U}$$

$$\Leftrightarrow \quad \frac{d\sigma^2}{dx} = 2\frac{K}{U}$$

This is the condition for the Gaussian plume equation to be a solution to Eq (1).

If $K =$ constant and $\sigma = 0$ at $x = 0$,

$$\sigma^2 = 2\frac{x}{U}K$$