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Qu 1.a) Since the water is incompressible conservation of mass says that the volume of water within each wavelength λ ($=2\pi/k$) must be unchanged by the perturbation i.e.

$$\pi a^2\lambda = \int_0^\lambda \pi r^2 dx = \pi\int_0^\lambda (\alpha+\beta\cos kx)^2 dx$$

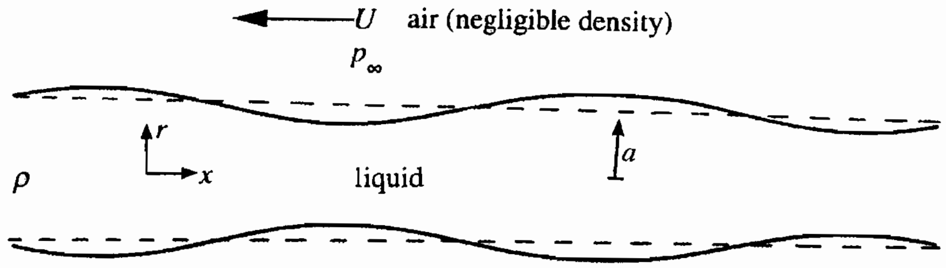
$$= \pi\int_0^\lambda (\alpha^2+2\alpha\beta\cos kx+\beta^2\cos^2 kx)\,dx = \pi\lambda(\alpha^2+\tfrac{1}{2}\beta^2).$$

Hence $\alpha^2 = a^2 - \frac{1}{2}\beta^2 = a^2\left(1-\frac{1}{2}\frac{\beta^2}{a^2}\right)$

$$\alpha = a\left(1-\frac{1}{2}\frac{\beta^2}{a^2}\right)^{1/2} \simeq \underline{\underline{a\left(1-\frac{1}{4}\frac{\beta^2}{a^2}\right)}} \quad (1)$$

after using the binominal theorem for small β^2. [30%]

b) Change reference frame so that the water is at rest and the air blows past it with speed U. Since the air has negligible density this removes kinetic energy from the problem.



Potential energy = $\sigma \times$ surface area.

Initial jet surface in a length $\lambda = 2\pi a\lambda$

Final jet surface in length $\lambda = \int_0^\lambda 2\pi r\,ds$ where $(ds)^2=(dx)^2+(dr)^2$

$$ds = dx\left(1+\left(\frac{dr}{dx}\right)^2\right)^{1/2} = dx\left(1+(-\beta k\sin kx)^2\right)^{1/2}$$

$$= dx\left(1+\tfrac{1}{2}\beta^2k^2\sin^2 kx\right)$$ after using the binominal theorem

After substituting for r and ds,

Final jet surface in length $\lambda = \int_0^\lambda 2\pi(\alpha+\beta\cos kx)(1+\frac{1}{2}\beta^2k^2\sin^2 kx)\,dx$

$$= 2\pi\int_0^\lambda (\alpha+\beta\cos kx+\alpha\tfrac{1}{2}\beta^2k^2\sin^2 kx+\tfrac{1}{2}\beta^3k^2\sin^2 kx)\,dx$$

$$= 2\pi\lambda\left(\alpha+\frac{\alpha}{4}\beta^2k^2\right) \text{ for small } \beta^2$$

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Qu 1 cont.)

After substituting for α in terms of a and β^2 from eqn.(1), we obtain,

$$\text{Final jet surface in length } \lambda = 2\pi\lambda a\left(1 - \frac{1}{4}\frac{\beta^2}{a^2} + \frac{1}{4}\beta^2 k^2\right)$$

Hence change in potential energy = $\sigma \times$ change in surface area

$$= \sigma \, \frac{2\pi\lambda a}{4}\left(k^2 - \frac{1}{a^2}\right)\beta^2$$

If $ka > 1$, the potential energy has increased

$ka < 1$ the potential energy has decreased.

Hence the jet is unstable for wavenumbers, $\underline{ka < 1}$. [30%]

b). The energy argument is exactly as for the water jet. The potential energy = $\sigma \times$ area and it does not matter whether the heavy liquid is inside or outside the jet. Hence again the unstable wavenumbers have $ka < 1$ [20%]

c). The forms of the linear distances are different for the cases when the jet is water or the air jet is in water. For the water jet, the solution of Laplace's equation for the velocity potential has to be finite on $r = 0$ and this involves the I_0 modified Bessel functions. For case b) Laplace's equation has to be solved in the water and we need a solution that remains finite as $r \to \infty$, the K_0 solution. These different solutions lead to different fastest growing wavenumbers and hence to different size droplets / bubbles. [20%]

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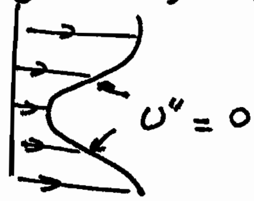
Qu 2 a) Rayleigh's inflexion point theorem for parallel shear flows states that a shear flow with profile $U(z)$ where U is in x (or y) direction is only inviscidly unstable if $\frac{d^2U}{dz^2} = 0$ for some z.

(i) there are a number of inviscidly unstable flows according to Rayleigh's inflexion point theorem, e.g. a wake flow

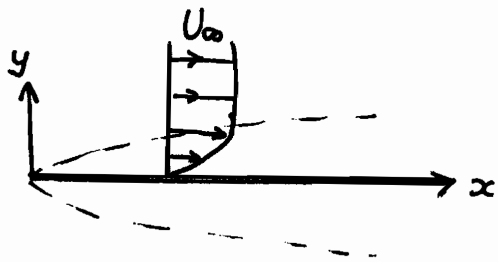


Other examples are mixing and jet flows and convection near a hot stationary wall.

(ii) a boundary layer flow does not have a inflexion point and is inviscidly stable. In this case viscosity is destabilising. [25%]

b). Spatial stability means investigating the development in space of a localised disturbance which is periodic in time. [5%]

c).



The Blasius profile for a laminar boundary layer is self-similar and the mean velocity profile can be written

$$\frac{U(x,y)}{U_\infty} = f\left(\frac{y}{\delta(x)}\right)$$

where the boundary layer thickness grows with distance from the leading edge like $x^{1/2}$.

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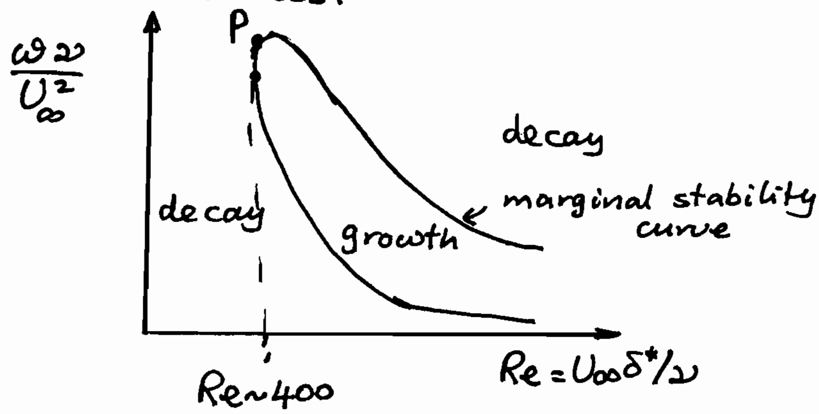
Qu 2 cont.)

Since the boundary layer is thin the disturbances that arise can be understood by considering linear perturbations to a parallel mean flow whose mean velocity profile is equal to the local mean velocity. For disturbances of frequency ω, this leads to growth or decay according to the local value of the Reynolds number $Re = U_\infty \delta^*/\nu$ where δ^* = displacement thickness.



Near the leading edge, disturbances of any frequency decay. At a particular distance x_1 downstream of the leading edge corresponding to $\delta^*(x_1) \sim \nu\, 400/U_\infty$ one frequency becomes unstable (position P on the curve). However a disturbance of this frequency travels downstream where the boundary layer is thicker and the disturbance decays. Further downstream a range of frequencies will be amplified and Tollmien-

Schlichting waves will be seen. Once the wave amplitude exceeds a critical value, nonlinear and three-dimensional effects become important. Complex flow structures appear which ultimately lead to turbulence.

[50%]

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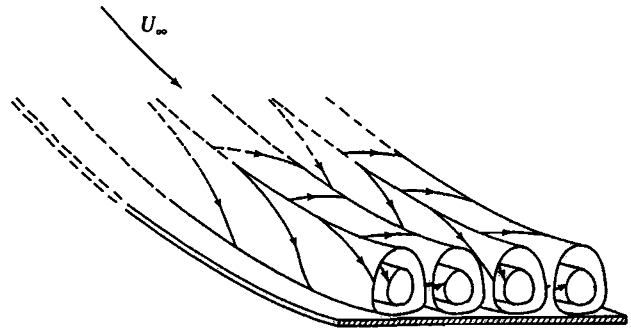
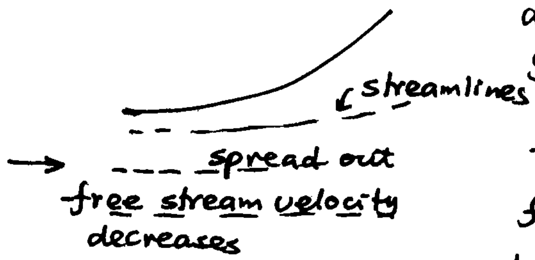
Qu 2 cont.)

d) When the plate is curved, on the concave side there could be a curved streamline instability leading to streamwise Görtler vortices.



On the convex side, the mean flow is slowed down and so there is an adverse pressure gradient. This will result in a somewhat thicker boundary layer.



The perturbations will be similar in form to those on the flat plate but because of the relatively thicker boundary layer instability will occur nearer to the leading edge. If the curvature is significant there is also the possibility of separation. [20%]

Qu 3a) $\phi = \frac{Ua^3}{2r^2}\cos\theta$

$$\frac{\partial\phi}{\partial t} = \frac{\partial U}{\partial t}\frac{a^3}{2r^2}\cos\theta \qquad \left.\frac{\partial\phi}{\partial t}\right|_{r=a} = \frac{\partial U}{\partial t}\frac{a}{2}\cos\theta$$

$$\frac{\partial\phi}{\partial r} = -\frac{Ua^3\cos\theta}{r^3} \qquad \left.\frac{\partial\phi}{\partial r}\right|_{r=a} = -U\cos\theta$$

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{Ua^3}{2r^3}\sin\theta \qquad \left.\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right|_{r=a} = -\cancel{1}\frac{U\sin\theta}{2}$$

$$f_x = \int_0^\pi p\, 2\pi a^2 \cos\theta\sin\theta\, d\theta$$

$$= \int_0^\pi -\rho 2\pi a^2\left[\frac{\partial U}{\partial t}\frac{a}{2}\cos\theta + \frac{1}{2}\left(U^2\cos^2\theta + \frac{1}{4}U^2\sin^2\theta\right)\right]\cos\theta\sin\theta\, d\theta$$

$$= \int_0^\pi -\rho 2\pi a^2\frac{\partial U}{\partial t}\frac{a}{2}\cos^2\theta\sin\theta\, d\theta + \int_0^\pi -\rho 2\pi a^2\left(\frac{U^2}{2}\cos^2\theta + \frac{U^2}{8}\sin^2\theta\right)\cos\theta\sin\theta\, d\theta$$

$$= -\rho\pi a^3\frac{\partial U}{\partial t}\underbrace{\int_0^\pi \cos^2\theta\sin\theta\, d\theta}_{①} - \cancel{\rho 2\pi}\;\rho\pi a^2U^2\underbrace{\int_0^\pi \left(\cos^3\theta\sin\theta + \frac{1}{4}\sin^3\theta\cos\theta\right)d\theta}_{②}$$

$\int$ ① & ② by sub

$a = \cos\theta$
$da = -\sin\theta\, d\theta$

$$① \quad \left[-\frac{\cos^3\theta}{3}\right]_0^\pi = \frac{2}{3}$$

$$② \quad \left[-\frac{\cos^4\theta}{4} + \frac{\sin^3\theta}{3}\right]_0^\pi = 0$$

$$\therefore f_x = -\frac{2}{3}\rho\pi a^3\frac{\partial U}{\partial t}$$ where ρ is density of fluid

b) - extra mass due to force required to accelerate the body AND fluid around it.
- accounts for the work done to change K.E. of fluid
- can be viewed conceptually as "entrained mass"
- only important when inertial terms dominate

added mass for sphere from a) $\frac{2}{3}\rho\pi a^3$

c) Initially it may appear that added mass terms equate to the volume of fluid displaced or some fraction of it, there is NO general correlation between added mass and displaced fluid. The case for cylinder is coincidental as discussed in lectures.

d) $m_{ball} = \frac{4}{3}\pi a^3 \rho_{air}$, $m_a = \frac{2}{3}\pi a^3 \rho_{H_2O}$

$$\frac{m_a}{m_{ball}} = \frac{660}{1.56} \simeq 417$$

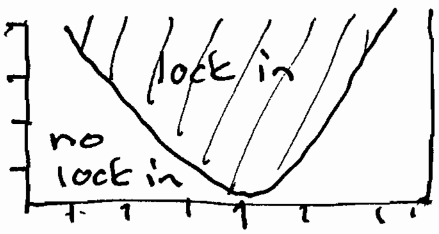
$\therefore$ added mass is over 400x greater than ping pong ball

Qu 4 a) Vortex shedding occurs when the flow separates from the sides of the cylinder and roll-up into vortices that form an oscillating wake downstream. Vortices are shed at a well defined frequency defined by the Strouhal no., $St = fD/U$ which is usually between 0.2-0.3.

The cylinder will oscillate from side to side at this vortex shedding frequency, f_s.

As the flow velocity is increased/decreased f_s is changed accordingly. If f_s approaches the natural resonance frequency of the cylinder, f_n, the shedding will "lock-in" to f_n and large amplitude oscillations may result.

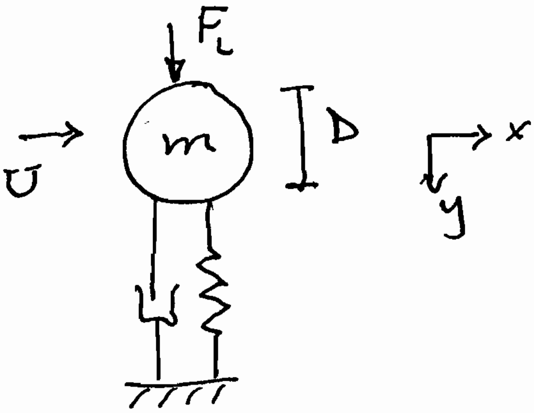
normalised amplitude. $\frac{A}{D}$



[30%]

f_s/f_n

b)



F_L - is force due to vortex shedding which is harmonic

$$F_L = \tfrac{1}{2}\rho U^2 D C_L \sin(\cancel{f_s}\,t)$$
ω_s

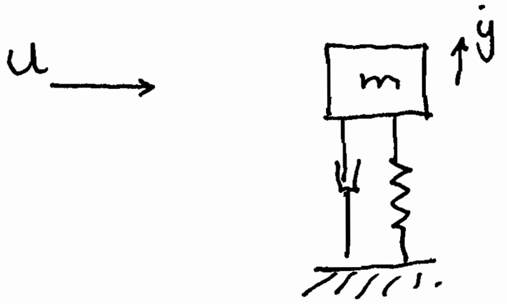
Mass-Spring-Damper has following eqtn of motion:

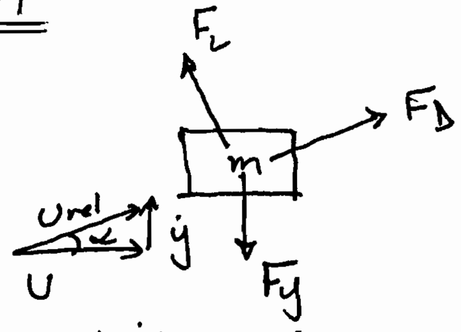
$$m\ddot{y} + \underbrace{2\zeta\omega_y\dot{y}}_{\text{damper}} + \underbrace{ky}_{\text{spring}} = F_L$$

$$\therefore\quad m\ddot{y} + 2\zeta\omega_y\dot{y} + ky = \tfrac{1}{2}\rho U^2 D C_L \sin(\omega_s t) \quad [30\%]$$

Qu 4 cont.)

c)





$\alpha = \tan^{-1}\left(\frac{\dot{y}}{U}\right) \approx \frac{\dot{y}}{U}$ as $\alpha \to 0$

Flow separates over beam in same way as before but unlike the cylinder, any displacement ($\dot{y}$) of the beam changes the apparent angle of attack, α.

Equation of motion is:

$$m\ddot{y} + 2m\zeta\omega_y\dot{y} + ky = F(t)$$

$F(t) = \frac{1}{2}\rho U^2 D c_y$ (U not Urel)

$c_y = \alpha \frac{\partial c_y}{\partial \alpha}$ for small α (α = angle of attack)

$= \frac{\dot{y}}{U}\frac{\partial c_y}{\partial \alpha}$

$$\Rightarrow \quad m\ddot{y} + 2m\zeta\omega_y\dot{y} + ky = \frac{\dot{y}}{U}\frac{\partial c_y}{\partial \alpha}\left(\frac{1}{2}\rho U^2 D\right)$$

collect damping terms

$$\Rightarrow \quad m\ddot{y} + \dot{y}\left[2m\zeta\omega_y - \frac{\partial c_y}{\partial \alpha}\left(\frac{1}{2}\rho U D\right)\right] + ky = 0$$

Galloping starts when: $\frac{1}{2}\rho U D \left.\frac{\partial c_y}{\partial \alpha}\right|_{\alpha=0} > 2m\zeta\omega_y$

[40%]