2009 PART IIB 4A12 TURBULENCE AND VORTEX DYNAMICS

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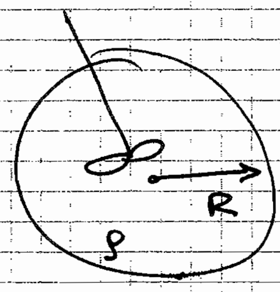
Q1 (a)



Power in = dissipation × mass

$$P = \frac{u^3}{L_{turb}} \cdot \rho \cdot \frac{4}{3}\pi R^3$$

$$\Rightarrow L_{turb} = \frac{4}{3}\pi R^3 \rho \frac{u^3}{P}$$

(b) $\frac{dk}{dt} = -\varepsilon \iff \frac{3}{2}\frac{du^2}{dt} = -\frac{u^3}{L_{turb}}$

$$\iff \frac{du^2}{dt} = -\frac{2}{3}\frac{1}{L_{turb}} u^3$$

with $\alpha = \frac{u}{u_o}$, $T_o = \frac{L_{turb}}{u_o}$, $\tau = \frac{t}{T_o}$ $\Big\}$, $\frac{d\alpha^2}{d\tau} = -\frac{2}{3}\alpha^3$

$$\iff \frac{d\alpha^2}{\alpha^3} = -\frac{2}{3}\tau$$

$$\iff \boxed{\frac{u}{u_o} = \left(1 + \frac{t}{3T_o}\right)^{-1}}$$

(c) Variance equation: $\frac{d\sigma^2}{dt} = -2\sigma^2 \frac{u}{L_{turb}}$

$$\iff \frac{d\sigma^2}{dt} = -2\frac{\sigma^2}{T_o}\frac{u}{u_o}$$

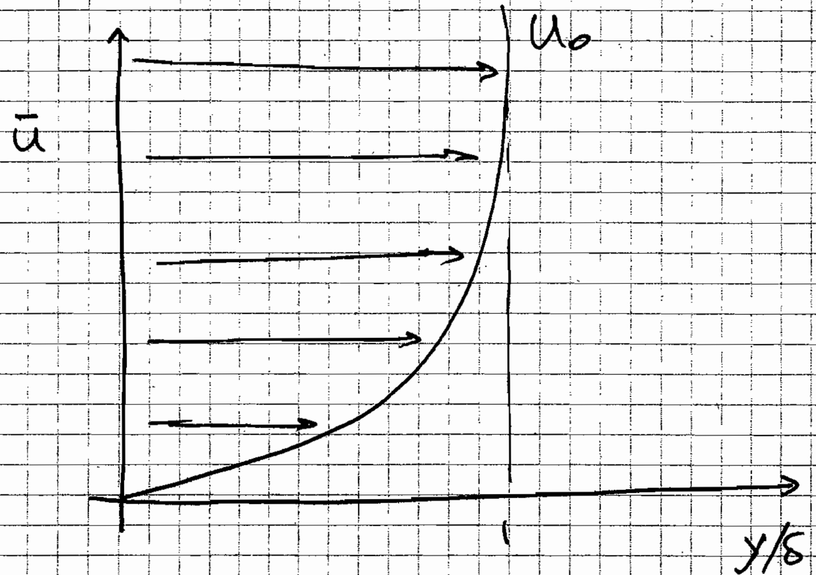
$x = \sigma^2/\sigma_o^2 \Rightarrow \frac{dx}{d\tau} = -2x\left(1 + \frac{\tau}{3}\right)^{-1}$

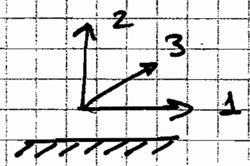
$$\Rightarrow \frac{dx}{x} = -\frac{2\,d\tau}{1 + \frac{\tau}{3}}$$

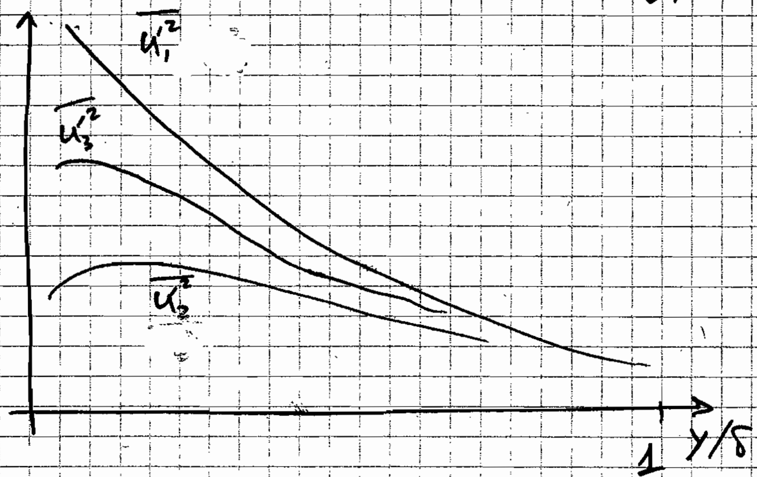
$$\Rightarrow \ln x = -6\ln\left(1 + \frac{\tau}{3}\right)$$

$$\Rightarrow \boxed{\sigma = \sigma_o\left(1 + \frac{t}{3T_o}\right)^{-3}}$$

Q2 (a)







The streamwise component $\overline{u_1'^2}$ is generated by shear. By the pressure interaction terms, the turbulence is re-distributed to $\overline{u_2'^2}$ & $\overline{u_3'^2}$.

(b) In the free turbulent layer, Prandtl assumed that the shear stress is negligible and that the turbulent stress $\overline{u_1' u_2'} = -\nu_{turb} \frac{\partial \bar{u}}{\partial y}$ with $\nu_{turb} = \kappa u^* y$. The advection terms are negligible $\Rightarrow$ $0 = \frac{\partial}{\partial y}\left(\kappa u^* y \frac{\partial \bar{u}}{\partial y} \right)$

$$\Rightarrow \kappa u^* y \frac{\partial \bar{u}}{\partial y} = u^{*2}$$ (the stress at the wall)

Q2 cont'd

Using $y^+ = y/(\nu/u^*)$, we can integrate to get

$$\frac{\bar{u}}{u^*} = \frac{1}{\kappa} \ln y^+ + A \qquad \text{Log-law of the wall}$$

Module 4A2 Exam Questions 2009 Cribs

3. (a) Simpler models may include eddy-diffusivity models or mixing-length models. Eddy-diffusivity models lack sufficient flexiblity since the diffusivity is fixed and has to be tuned a priori for one situation. Similarly Mixing length models, while allowing greater flexibility through at least allowing k to vary, suffer from a fixed length scale. Two-equation models ($k-\varepsilon$), also $k-\omega$, allow both k and length scale to vary according to the local flow conditions.

(b) 1) length scale is too large in near wall flows
Boundary-layer flows: use a low Re modification such as a wall-function.

2) Excessive production of k in impinging flows
Impinging-jet or stagnation-point flows: use a Kato-Launder or similar correction.

3) Flows with strong streamline curvature have too much dissipation. Recirculating flows have short recirculation zones: use a Reynolds-stress model

4) Round jets have low spreading rate. In a round jet flow, use a correction such as Pope's term.

4(a) $$\frac{\partial k}{\partial t} + U_k \frac{\partial k}{\partial x_k} = \frac{\partial}{\partial x_k} \Gamma_k \frac{\partial k}{\partial x_k} + P_k - \varepsilon$$

$$\frac{\partial \varepsilon}{\partial t} + U_k \frac{\partial \varepsilon}{\partial x_k} = \frac{\partial}{\partial x_k} \Gamma_\varepsilon \frac{\partial \varepsilon}{\partial x_k} + C_{\varepsilon 1} \frac{\varepsilon P_k}{k} - C_{\varepsilon 2} \frac{\varepsilon^2}{k}$$

Homogeneous isotropic turbulence : all spatial gradients are zero.

$$P_k = \mu_t \frac{\partial u_i}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = 0$$

Equations become

$$\frac{\partial k}{\partial t} = -\varepsilon \quad ; \quad \frac{1}{k}\frac{\partial k}{\partial t} = -\varepsilon/k$$

$$\frac{\partial \varepsilon}{\partial t} = -C_{\varepsilon 2}\,\varepsilon^2/k \quad ; \quad \frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial t} = -C_{\varepsilon 2}\,\varepsilon/k$$

$$\therefore \quad \frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial t} = C_{\varepsilon 2}\frac{1}{k}\frac{\partial k}{\partial t} \Rightarrow \ln \varepsilon = C_{\varepsilon 2} \ln k + \text{const}$$

$k = k_0$ when $\varepsilon = \varepsilon_0$

$\therefore \text{const} = \ln \varepsilon_0 - C_{\varepsilon 2} \ln k_0$

$\therefore$ have $(k/k_0)^{C_{\varepsilon 2}} = \varepsilon/\varepsilon_0$ and $\varepsilon = \varepsilon_0 (k/k_0)^{C_{\varepsilon 2}}$

Then $$\frac{\partial k}{\partial t} = -\varepsilon_0 (k/k_0)^{C_{\varepsilon 2}}$$

$$\frac{1}{k^{C_{\varepsilon 2}}}\frac{k_0^{C_{\varepsilon 2}}}{k_0}\frac{\partial k}{\partial t} = -\varepsilon_0/k_0 \Rightarrow \frac{(k/k_0)^{1-C_{\varepsilon 2}}}{1-C_{\varepsilon 2}} = -\varepsilon_0/k_0\, t + \text{const}$$

at $t=0$, $k=k_0$ $\therefore$ $\text{const} = \dfrac{1}{1-C_{\varepsilon 2}}$

$$\therefore \quad (k/k_0)^{1-C_{\varepsilon 2}} = (1-C_{\varepsilon 2})\left(1 - \varepsilon_0/k_0 \, t\right)$$

$$\therefore \quad k/k_0 = (1-C_{\varepsilon 2})^{\frac{1}{1-C_{\varepsilon 2}}} \left(1 - \frac{\varepsilon_0 t}{k_0}\right)^{\frac{1}{1-C_{\varepsilon 2}}}.$$

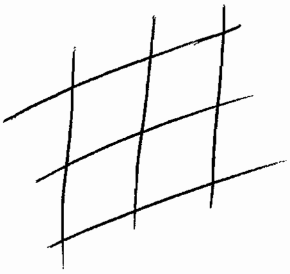
ie $\quad k \sim t^{\frac{1}{1-C_{\varepsilon 2}}}$

Value of $C_{\varepsilon 2} \sim 1.92$ $\quad \therefore \quad k \sim t^{-1/2}$ approx.

$$\varepsilon/\varepsilon_0 = (k/k_0)^{C_{\varepsilon 2}}$$

$$\therefore \quad \varepsilon/\varepsilon_0 = (1-C_{\varepsilon 2})^{\frac{C_{\varepsilon 2}}{1-C_{\varepsilon 2}}} \left(1 - \frac{\varepsilon_0}{k_0} t\right)^{\frac{C_{\varepsilon 2}}{1-C_{\varepsilon 2}}}$$

$\therefore \quad \varepsilon \sim t^{C_{\varepsilon 2}/1-C_{\varepsilon 2}}$ $\quad \therefore \quad \varepsilon \sim t^{-2}$ approx.

(b) 

Wind tunnel turbulence – uniform flow through a grid.

k can be measured with hot wire. This gives $\overline{u_1'^2}$, but assuming local isotropy we estimate k.

ε can be measured using $\varepsilon = 15 \dfrac{\nu}{\overline{u}_1^2} \overline{\left(\dfrac{\partial u_1'}{\partial t}\right)^2}$

assuming Taylor's hypothesis.