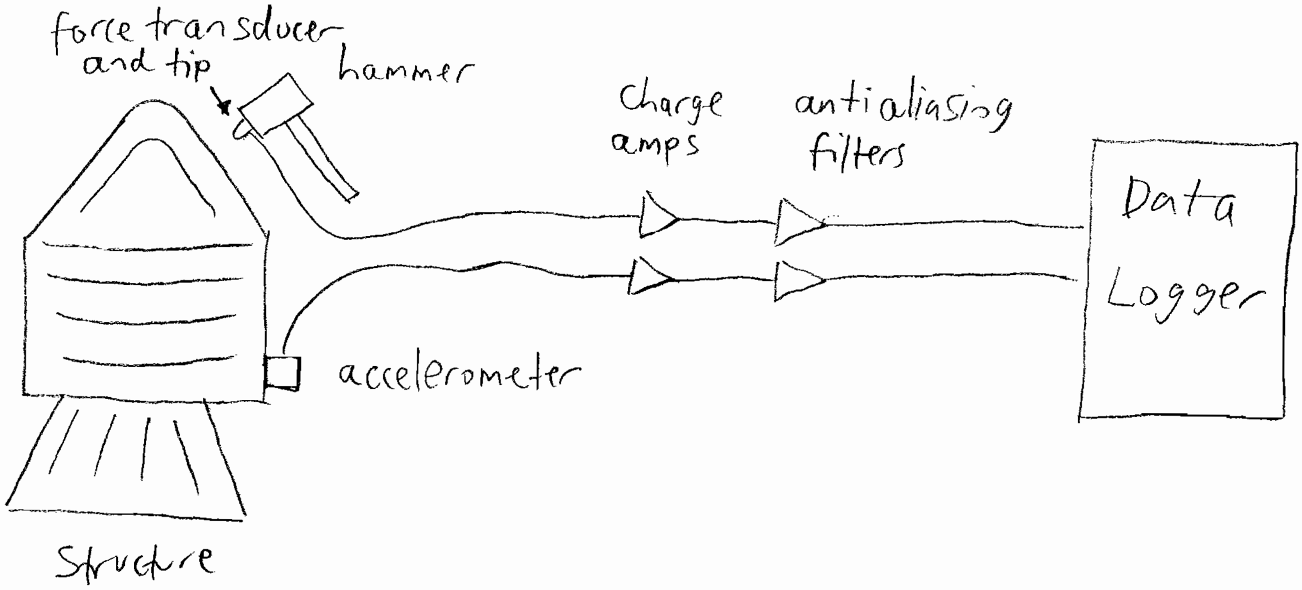
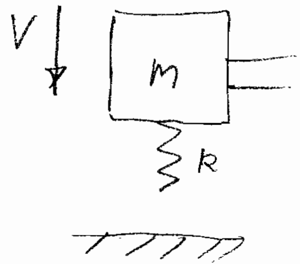
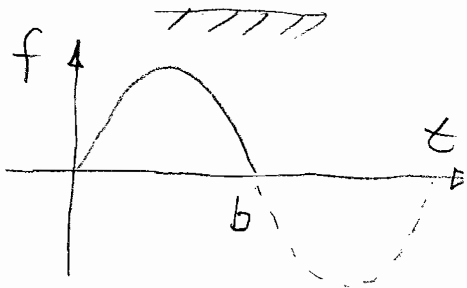
1(a)



(i) Hammer mass m, tip stiffness k produces a half-sine force pulse of duration $b = \pi\sqrt{\frac{m}{k}}$



So choose a soft tip to generate a long pulse. A Pulse of duration b excites frequencies up to $\sim \frac{1}{b}$ Hz (rule of thumb)

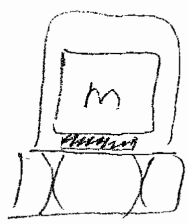


Peak force : $\frac{1}{2}mV^2 = \frac{1}{2}kx^2 \quad \therefore \quad (kx)^2 = mkV^2$

$$F_{peak} = kx = \sqrt{mk}\,V = \frac{k}{\pi}bV$$

So: select b for desired frequency range
select m & k for desired peak force

(ii) An accelerometer works by measuring the force required to accelerate a mass m, using $F = ma$

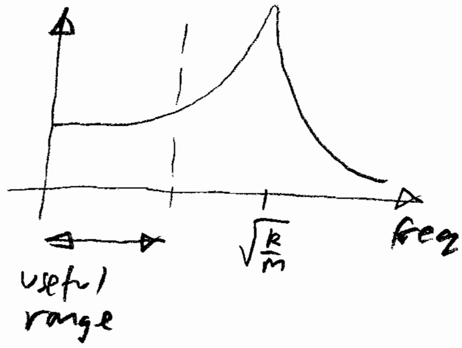


(ii) cont So a large output needs a large mass hence high-sensitivity accelerometers are heavy

Tradeoff:

the piezo force element has a stiffness k so the accelerometer has a resonance at $\sqrt{\frac{k}{m}}$

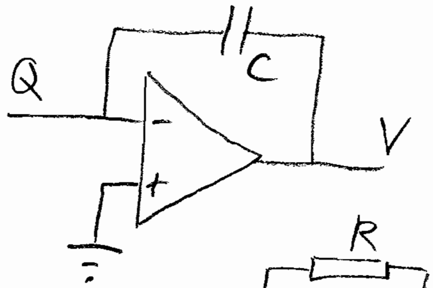
This means that heavy high-sensitivity accelerometers have a small useful frequency range.



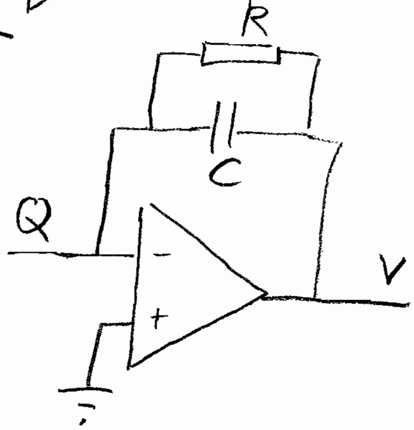
But these days a tiny accelerometer can be constructed with low-noise electronics (low noise floor)
Quite possibly the days of heavy accelerometers are over.

(iii) Force transducer and accelerometers produce tiny charge outputs (pico Coulombs). Typical charge amp gain $\sim 25\ \frac{mV}{pC}$

$$V = -\frac{Q}{C}$$

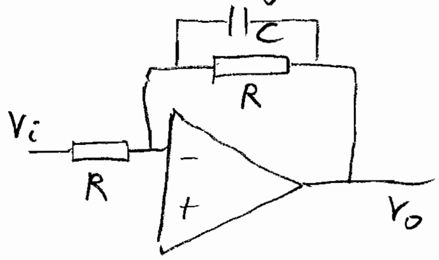


But to avoid buildup of charge on C need a bleed resistor R which acts as a high-pass filter at frequency $\frac{1}{2\pi}\sqrt{\frac{1}{RC}}$ Hz

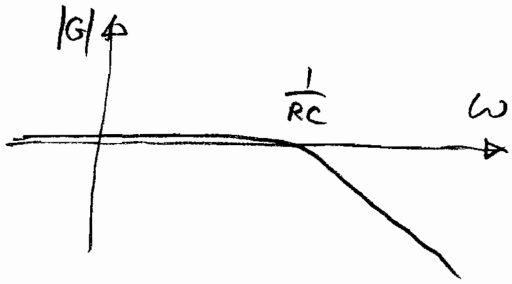


(iv) The low-pass filter is for "anti-aliasing".

It might look like this :



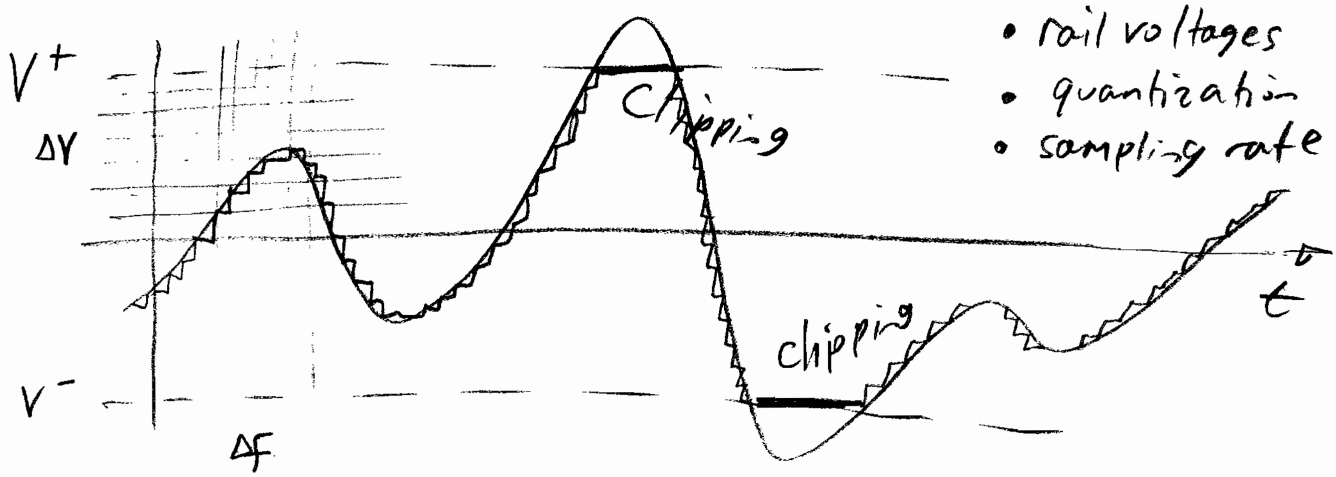
$$G = \frac{V_o}{V_i} = - \frac{1}{1 + j\omega RC}$$



It is important to ensure that the logged data has no content above the Nyquist frequency $= \frac{f_{sample}}{2}$

(v) The data logger has three key parameters

- rail voltages
- quantization
- sampling rate



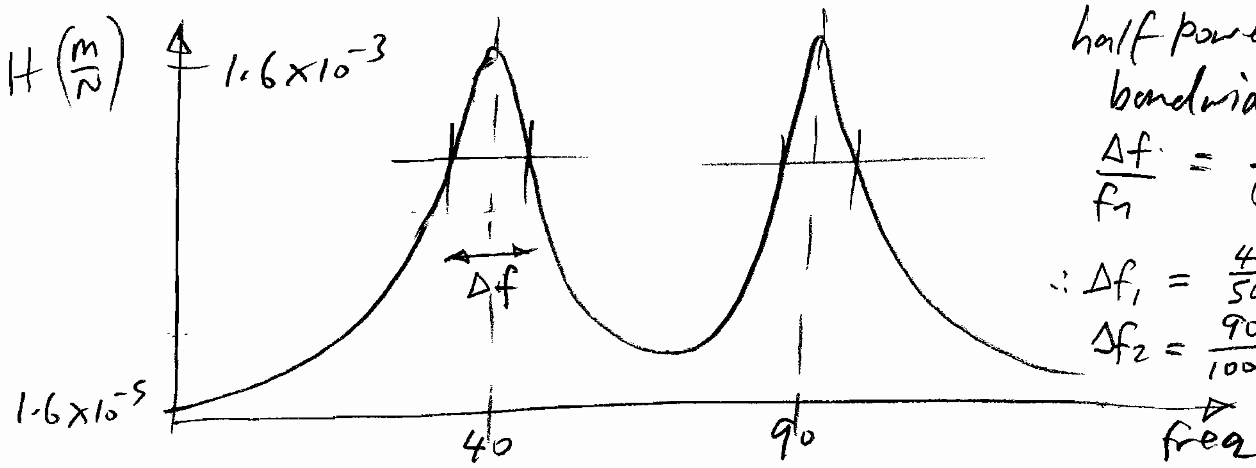
The acquired data has discrete steps in both the voltage and time directions. Also the signal is truncated at the rail voltages V^+ & V^- known as "clipping".

(b) At resonance, $H = \dfrac{u_j^{(n)} u_k^{(n)}}{2 i \omega_n^2 \zeta_n} = Q_n \dfrac{u_j u_k}{\omega_n^2}$

$\zeta_n = \dfrac{1}{2Q}$ (data book)

mode 1 $\quad |H_{peak}| \simeq 50 \dfrac{2}{(2\pi\, 40)^2} = 0.0016 \dfrac{m}{N}$

mode 2 $\quad |H_{peak}| \simeq 100 \dfrac{5}{(2\pi\, 90)^2} = 0.0016 \dfrac{m}{N}$

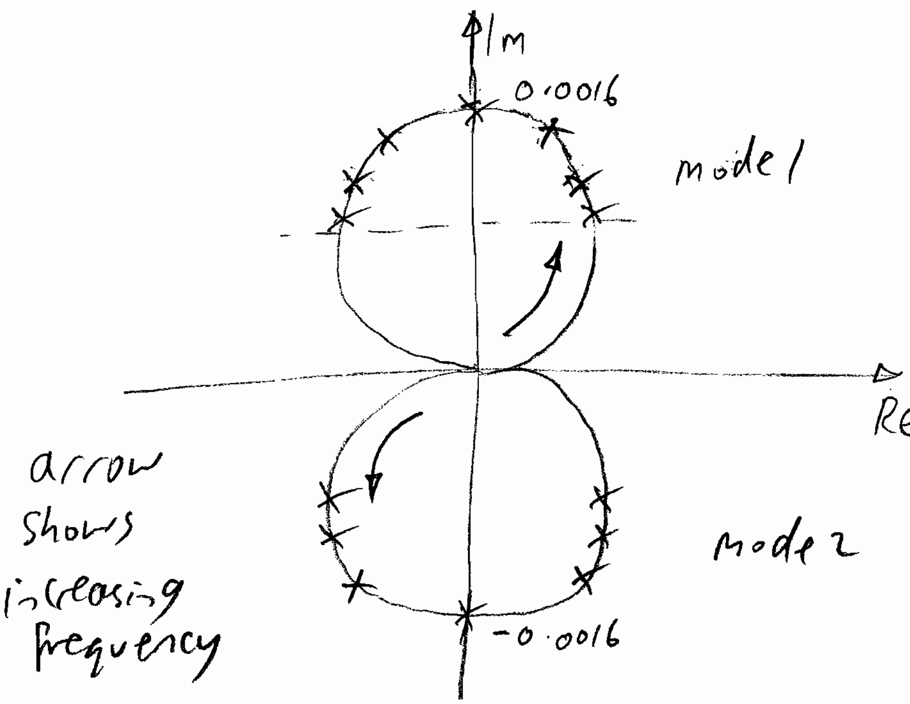


half power bandwidth $\dfrac{\Delta f}{f_1} = \dfrac{1}{Q}$

$\therefore \Delta f_1 = \dfrac{40}{50} = 0.8$

$\Delta f_2 = \dfrac{90}{100} = 0.9$

note, $\omega = 0 \quad |H| = \left| \dfrac{2}{\omega_1^2} - \dfrac{5}{\omega_2^2} \right| = 1.6\times10^{-5}$ m/N



Sampling rate = 1000 Hz

$N = 8192$

$\therefore \Delta f = \dfrac{1000}{8192} = 0.122$ Hz

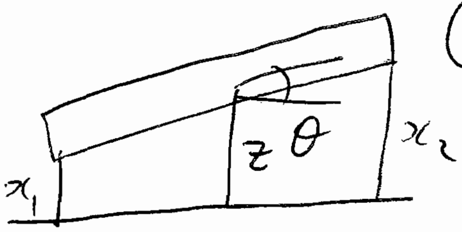
$\therefore$ approx 7 points in half power region

5

2. (a) Damping could be measured by:
(1) forced vibration, e.g. cyclic test in a tensile or torsion test rig;
(2) free vibration and measure in the frequency domain to obtain bandwidths of peaks, e.g. by circle fitting from a transfer function;
(3) free vibration and measure the decay times of individual modes.
(3) is best for very low damping (long decay is easy to measure). (2) is good for moderate damping, around 10^{-2}: peaks clearly visible, but not too narrow to get accurate bandwidth. (1) is best for high damping: if there are no resonances, phase lag or energy dissipation in forced vibration is the only option.

(i) Open-cell foam is likely to have moderate to high damping, to be lightweight, and to be hard to grip without damage or significant distortion. Could use forced vibration, or may just be possible at low frequencies to use free motion of a slab or beam specimen supported at nodal points, and measure the bandwidth of low resonances. For forced vibration need care in attaching the specimen to the machine: perhaps glue metal plates to the two faces of a block, then use shear, torsion or compression cyclic test. Air pumping through the cells will contribute to the damping: have to *decide* whether this should be included in the measurement of "material damping", or be treated as a separate effect. If the latter, need to measure in a vacuum. For free vibration tests, adding sensors may be problematic, so optical shadowing, laser vibrometer or a microphone might be good choices. In a cyclic forced vibration this is less of a problem, and the measurement would be part of the test machine.

(ii) Sheet glass will have very low damping but would be very sensitive to small cracks. The main problem for a measurement would be to support the specimen without adding a lot of additional damping from boundary effects. A free-free beam supported on thin rubber bands carefully placed under node lines would be a possible method. Impulsive excitation could be used, and the time decay measured, preferably using non-contact sensors like a laser vibrometer. Would need to put an opaque reflective spot on the specimen to get a laser reflection. Care is needed in making the specimen, to avoid microcracks at the edges from cutting.



2(b) $x_1 \simeq z - L\theta$, $x_2 \simeq z + L\theta$

So potential energy

$$V = \tfrac{1}{2}k(z-L\theta)^2 + \tfrac{1}{2}k(z+L\theta)^2$$
$$= k(z^2 + L^2\theta^2)$$

Kinetic energy $T = \tfrac{1}{2}m\dot{z}^2 + \tfrac{1}{2}I\dot{\theta}^2$

So $K = \begin{bmatrix} 2k & 0 \\ 0 & 2L^2k \end{bmatrix}$, $M = \begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}$

Both diagonal, so modes are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (bounce, pitch)

From Rayleigh quotient: $\omega_1^2 = \dfrac{2k}{m}$, $\omega_2^2 = \dfrac{2L^2k}{I}$

Now with damping V becomes $k(1+i\eta)(z^2+L^2\theta^2)$

So Rayleigh quotient: $\omega^2 \simeq \dfrac{2k(1+i\eta)(z^2+L^2\theta^2)}{mz^2 + I\theta^2}$

For small damping, OK to use undamped modes as approximate trial functions.

For $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$: $\omega^2 \simeq \dfrac{2k(1+i\eta)}{m} = \omega_1^2(1+i\eta)$

For $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$: $\omega^2 \simeq \dfrac{2kL^2(1+i\eta)}{I} = \omega_2^2(1+i\eta)$

So both damped modes have the same modal loss factor η, and hence Q factor $\dfrac{1}{\eta}$.

3 (a) Try $u(r,\theta,t) = f(r)\,g(\theta)\,e^{i\omega t}$

Substitute in the given equation:

$$T\left(f''g + \frac{1}{r}f'g + \frac{1}{r^2}fg''\right) = -m\omega^2 fg$$

Now need to rearrange to get all r's on the LHS, all θ's on the RHS. Can achieve this by multiplying through by $\frac{r^2}{fg}$.

Then
$$\underbrace{T\left(r^2\frac{f''}{f} + r\frac{f'}{f}\right) + m\omega^2 r^2}_{r \text{ only}} = \underbrace{-T\frac{g''}{g}}_{\theta \text{ only}}$$

So both sides must be constant: call the constant $n^2 T$ for convenience.

Then θ equation is $-\frac{g''}{g} = n^2$, or $g'' = -n^2 g$

with the general solution $\theta = A\sin n\theta + B\cos n\theta$

So n must be an integer, for θ to join up after one circuit. So $n = 0, 1, 2, 3, \ldots$

Now the r equation is $r^2 f'' + r f' + \left(\frac{m\omega^2}{T}r^2 - n^2\right) f = 0$

(b) Make a change of variable $z = r\sqrt{\frac{m\omega^2}{T}} = kr$ say.

Then $r^2 f'' \to z^2 \frac{d^2 f}{dz^2}$, $r f' \to z\frac{df}{dz}$, and

$$\frac{m\omega^2 r^2}{T} \Rightarrow z^2$$

So the equation becomes the standard form of the Bessel equation as given, so solutions are $f = J_n(z) = J_n(kr)$

To find natural frequencies, need to impose the boundary condition $u = 0$ at $r = a$,
so $f(a) = 0$, so $J_n(ka) = 0$

So each zero of the relevant Bessel function gives a possible value of k, and hence a possible value of ω.
Looking at the graphs, the function with a zero at the lowest value of z is J_0, with a zero at about $z = 2.5$ (exact answer is 2.404)
So the lowest natural frequency comes from $ka \simeq 2.5$

ie $a\omega_1\sqrt{\frac{m}{T}} \simeq 2.5$, or $\omega_1 \simeq \frac{2.5}{a}\sqrt{\frac{T}{m}}$

(c) For large z, $J_n(z) \to \sqrt{\frac{2}{\pi z}} \cos\left[z - \frac{(2n+1)\pi}{4}\right]$

So the zeros occur where

$$z - \frac{(2n+1)\pi}{4} \simeq (m + 1/2)\pi$$

for any integer value of m.

So expect natural frequencies at

$$\omega \simeq \frac{1}{a}\sqrt{\frac{T}{m}}\left[(m + 1/2)\pi + \frac{(2n+1)\pi}{4}\right]$$

$$= \frac{1}{a}\sqrt{\frac{T}{m}}\left[m\pi + \frac{n\pi}{2} + \frac{3\pi}{4}\right]$$

Note that these m values will <u>not</u> correctly number the modes from the lowest – we can't tell from the asymptotic formula how many natural frequencies occur at low frequencies.

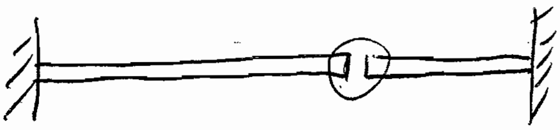
4 (a)(i) Adding a point mass can be regarded as coupling two systems at a point — the original system and an isolated mass, which has a "natural frequency" of zero.
So the new natural frequencies must interlace the old frequencies together with $\omega = 0$. In other words, all frequencies move <u>down</u> (or possibly some stay the same) when the mass is added.

(ii) Moving a mass does not change the number of degrees of freedom: we gain one DoF by detaching the mass, then lose it again when the mass is re-attached. Each step separately can be described by the interlacing theorem: Frequencies interlace <u>upwards</u>, then <u>downwards</u>

Original ω_ns : × × × × ×
Remove mass : o o o o
Put it back : + + + + +

Final frequencies can't be more than 2 places away in the original mode series, but they need not interlace strictly: can have 2, 1 or 0 modes in each gap, but average density of modes stays the same

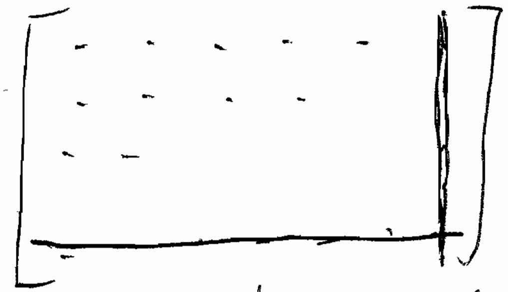
(iii)



Welding together the tips imposes 2 constraints: displacement and rotation both match. If added one at a time (free → pinned → welded), get interlacing upwards at each stage. Final results need not interlace, but similar pattern to case (ii): no more than 2 places shift, 2/1/0 modes in each gap, same density.

(b) We can number the degrees of freedom any way we like, so suppose the one we choose to constrain is the last one, q_N. The reduced system has $N-1$ degrees of freedom, and the new potential and kinetic energy expressions are the same as the old ones but with $q_N = 0$, $\dot{q}_N = 0$.

So the new M & K are found by deleting the last row and column from the old matrices



So the eigenvalues of a matrix with the last row & column deleted must interlace those of the original matrix. Same happens again if we delete a further row & column.
So a numerical strategy could be: start with the 2×2 matrix in the top LH corner, & find its eigenvalues from the quadratic formula.
Now add 3rd row & column back. 3×3 matrix must have 1 eigenvalue below old λ_1, one in the gap, and one above λ_2. Knowing this, an efficient search can be made for each in turn, e.g. by interval halving. Now add the 4th row & column and search again, and so on until the whole matrix has been restored.

Engineering Tripos Part IIB 2009

4C6: Advanced linear vibration

Answers

2. Damped frequencies $\sqrt{\dfrac{2k(1+i\eta)}{m}}$, $\sqrt{\dfrac{2kL^2(1+i\eta)}{I}}$

3(c). $\omega \approx \dfrac{1}{a}\sqrt{\dfrac{T}{m}}\left[k\pi + \dfrac{n\pi}{2} + \dfrac{3\pi}{4}\right]$, $k = 1,2,3\ldots$ (but only valid for large k)