2009 PART IIB 4C7 RANDOM AND NON-LINEAR VIBRATIONS
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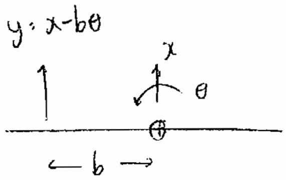
1. a)




$$y = x - b\theta \Rightarrow H_{y\eta} = H_{x\eta} - b H_{\theta\eta}$$

$$\Rightarrow \underline{H_{y\eta} = \alpha\omega - i b\beta/\omega}$$ [10%]

b) For acceleration $H_{\ddot{y}\eta} = -\omega^2 H_{y\eta}$

$$S_{\ddot{y}\ddot{y}}(\omega) = |H_{\ddot{y}\eta}|^2 S_{\eta\eta}(\omega) = \omega^4 [\alpha^2\omega^2 + b^2\beta^2/\omega^2] S_{\eta\eta}(\omega)$$

$$\sigma_{\ddot{y}}^2 = S_0 \int_{\omega_1}^{\omega_2} \omega^4 [\alpha^2\omega^2 + b^2\beta^2/\omega^2] \, d\omega$$

$$\underline{\sigma_{\ddot{y}}^2 = \frac{S_0\alpha^2}{7}(\omega_2^7 - \omega_1^7) + \frac{S_0 b^2\beta^2}{3}(\omega_2^3 - \omega_1^3)}$$

For $\dddot{y}$, $H_{\dddot{y}\eta} = -i\omega^3 H_{y\eta} \Rightarrow S_{\dddot{y}\dddot{y}}(\omega) = \omega^2 S_{\ddot{y}\ddot{y}}(\omega)$

$$\Rightarrow \underline{\sigma_{\dddot{y}}^2 = \frac{S_0\alpha^2}{9}(\omega_2^9 - \omega_1^9) + \frac{S_0 b^2\beta^2}{5}(\omega_2^5 - \omega_1^5)}$$ [40%]

c) For $S_0 = 30$, $\alpha = 0.4$, $\beta = 4\times10^{-3}$, $\omega_1 = 0.3$, $\omega_2 = 0.8$ then :-

$$\sigma_{\ddot{y}}^2 = \frac{30\times0.4^2}{7}(0.8^7 - 0.3^7) + \frac{30\times50^2\times(4\times10^{-3})^2}{3}(0.8^3 - 0.3^3)$$

$$\sigma_{\ddot{y}}^2 = \underbrace{0.144}_{\text{from heave}} + \underbrace{0.194}_{\text{from pitch}} = \underline{0.338} \Rightarrow \underline{\sigma_{\ddot{y}} = 0.581} \text{ m/s}^2$$

Similarly : $\sigma_{\dddot{y}}^2 = \underbrace{0.0715}_{\text{from heave}} + \underbrace{0.07806}_{\text{from pitch}} = \underline{0.149} \Rightarrow \underline{\sigma_{\dddot{y}} = 0.386 \text{ m/s}^3}$

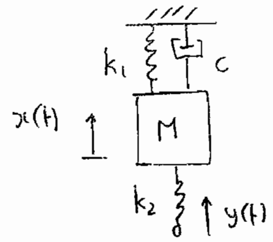
Probability of crossing $b$ : $1 - e^{-\nu_b^+ T}$

$$\nu_b^+ = \left(\frac{1}{2\pi}\right)\left(\frac{\sigma_{\dddot{y}}}{\sigma_{\ddot{y}}}\right) e^{-\frac{1}{2}\left(\frac{b}{\sigma_{\ddot{y}}}\right)^2}$$ with $b = 0.25\times10$

$$\nu_b^+ = \frac{1}{2\pi}\left(\frac{0.386}{0.581}\right) e^{-\frac{1}{2}\left(\frac{0.25\times 10}{0.581}\right)^2} = 1\times 10^{-5}$$

$$P = 1 - e^{-1\times 10^{-5}\times 3\times 60\times 60} = \underline{0.1} \Rightarrow 10\%$$ probability of exceedance. [50%]

2. a)




$$M\ddot{x} + C\dot{x} + k_1 x + k_2(x-y) = 0$$

$$M\ddot{x} + C\dot{x} + (k_1+k_2)x = k_2 y$$

$$\ddot{x} + \underbrace{2\beta\omega_n}_{C/M}\dot{x} + \underbrace{\omega_n^2}_{(k_1+k_2)/M} x = (k_2/M)y$$

Take F.T. $\Rightarrow$ $(-\omega^2 + 2i\beta\omega_n\omega + \omega_n^2)\, X(\omega) = (k_2/M)\, Y(\omega)$

$$X(\omega) = \underbrace{\left[\frac{k_2/M}{-\omega^2 + 2i\beta\omega_n\omega + \omega_n^2}\right]}_{H(\omega)} Y(\omega)$$

$$\underline{S_{xx}(\omega) = |H(\omega)|^2 = \left[\frac{(k_2/M)^2}{(\omega_n^2-\omega^2)^2 + (2\beta\omega_n\omega)^2}\right] S_{yy}(\omega)}$$ [25%]

b) Standard double-sided white noise result for $F = k_2 y/M$: $\sigma_x^2 = \dfrac{\pi S_0}{2\beta\omega_n^3}$ $\leftarrow$ $S_{FF}(\omega)$, in this case $(k_2/M)^2 S_0$

$$\Rightarrow \sigma_x^2 = \frac{\pi(k_2/M)^2 S_0}{2\beta\omega_n^3} = \frac{\pi(k_2/M)^2 S_0}{(C/M)(k_1+k_2)/M} = \underline{\frac{\pi k_2^2 S_0}{C(k_1+k_2)}}$$

$$\sigma_{\dot{x}}^2 = \omega_n^2\sigma_x^2 = \pi\left(\frac{k_1+k_2}{M}\right)\left(\frac{k_2^2 S_0}{C(k_1+k_2)}\right) = \underline{\frac{\pi k_2^2 S_0}{MC}}$$ [25%]

c) For white noise, $\sigma_x^2$ is inversely proportional to stiffness (in this case $k_1+k_2$). The mean squared velocity is $\omega_n^2 \times$ this result, to give a result independent of stiffness. [15%]

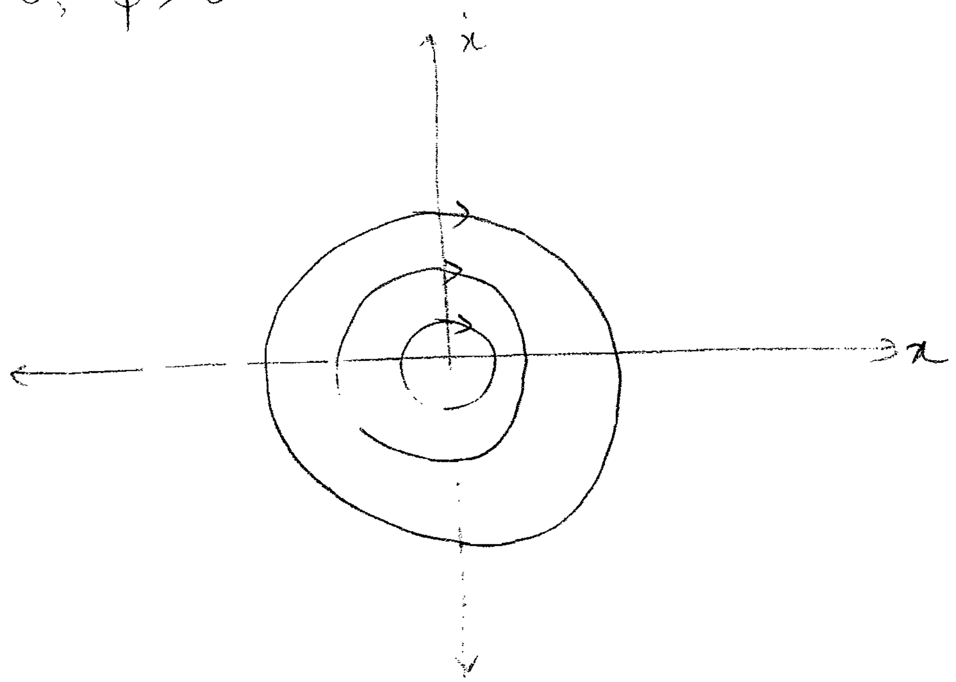
d) P = Power = damper force × velocity = $C\dot{x} \times \dot{x}$ = $C\dot{x}^2$

$$\Rightarrow E[P] = CE[\dot{x}^2] = C\sigma_{\dot{x}}^2 = \frac{C\pi k_2^2 S_0}{MC} = \underline{\frac{\pi k_2^2 S_0}{M}}$$ [35%]

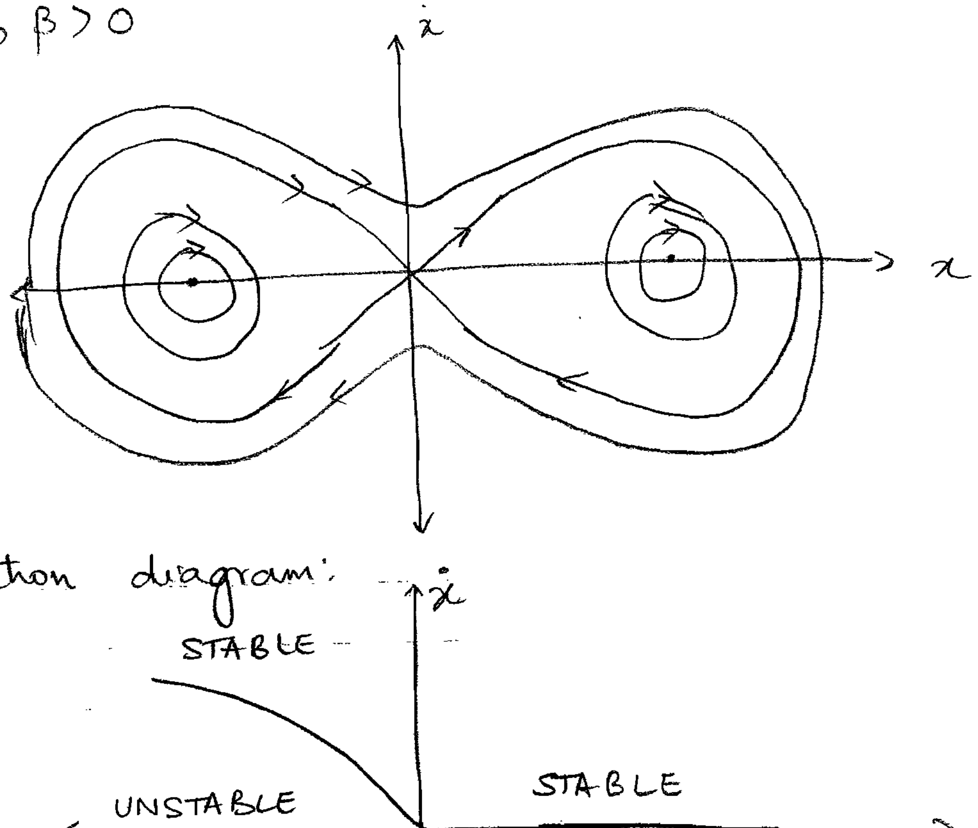
High damping $\Rightarrow$ low velocity, <u>but also</u> high coefficient $C$; effects cancel out to give a result independent of $C$. Does <u>not</u> happen for general $S_{yy}(\omega)$.

3

(a) $\alpha > 0$, $\beta > 0$

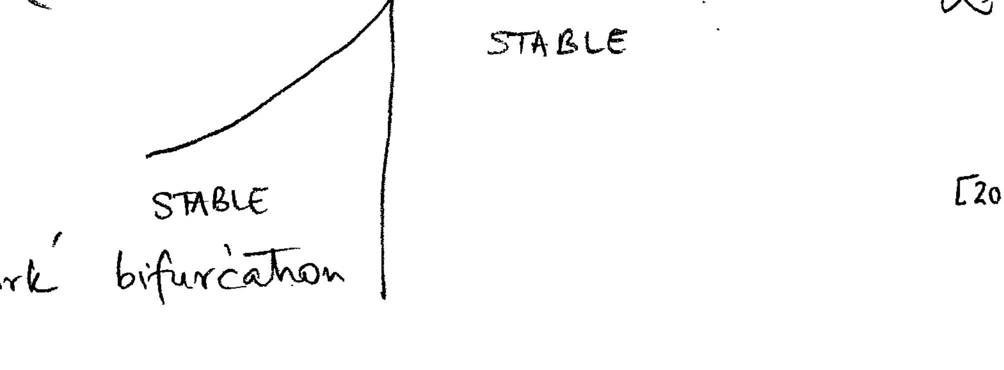



$\alpha < 0$, $\beta > 0$




[20%]

(b) Bifurcation diagram:




[20%]

∴ 'Pitchfork' bifurcation

(c) Nonlinear term is small Solve using iteration

$$\ddot{x} + \alpha x = -\beta x^3 + F_1 \cos\Omega_1 t + F_2 \cos\Omega_2 t$$

ZEROTH ORDER:

$$\ddot{x}_0 + \alpha x_0 = F_1 \cos\Omega_1 t + F_2 \cos\Omega_2 t$$

$$x_0 = \underbrace{\left(\frac{F_1}{\alpha - \Omega_1^2}\right)}_{G_1} \cos\Omega_1 t + \underbrace{\left(\frac{F_2}{\alpha - \Omega_2^2}\right)}_{G_2} \cos\Omega_2 t$$

FIRST ORDER

$$\ddot{x}_1 + \alpha x_1 = -\beta x_0^3 + F_1 \cos\Omega_1 t + F_2 \cos\Omega_2 t$$

$$\begin{aligned} &= F_1 \cos\Omega_1 t + F_2 \cos\Omega_2 t \\ &\quad + H_1 \cos\Omega_1 t + H_2 \cos\Omega_2 t \\ &\quad + H_3 \left[\cos(2\Omega_1 + \Omega_2)t + \cos(2\Omega_1 - \Omega_2)t\right] \\ &\quad + H_4 \left[\cos(\Omega_1 + 2\Omega_2)t + \cos(\Omega_1 - 2\Omega_2)t\right] \\ &\quad + H_5 \cos 3\Omega_1 t + H_6 \cos 3\Omega_2 t \end{aligned}$$

$$H_1 = -\frac{3}{4}\beta G_1 (G_1^2 + 2G_2^2) \; ; \; H_2 = -\frac{3}{4}\beta G_2 (2G_1^2 + G_2^2)$$

$$H_3 = -\frac{3}{4}\beta G_1^2 G_2 \; ; \; H_4 = -\frac{3}{4}\beta G_1 G_2^2$$

$$H_5 = -\frac{1}{4}\beta G_1^3 \; ; \; H_6 = -\frac{1}{4}\beta G_2^3$$

The solution $x_1$ will contain harmonic components @ $\Omega_1, \Omega_2, 2\Omega_1 \pm \Omega_2, \Omega_1 \pm 2\Omega_2, 3\Omega_1, 3\Omega_2$

In addition to harmonics of $\Omega_1$ and $\Omega_2$, the output also comprises of combinations of harmonics of $\Omega_1$ and $\Omega_2$. [60%]

4.

$$\ddot{x} + \varepsilon \sin \dot{x} + x = 0$$

(a) Transform to 2 1st order differential equations to work out equilibrium points

$$\dot{x} = y$$
$$\dot{y} = -\varepsilon \sin y - x$$

Equilibrium points when $\dot{x} = \dot{y} = 0$

$\Rightarrow x = 0, \dot{x} = 0$ is an equilibrium point

(b) Linearising about $x = 0, \dot{x} = 0$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -\varepsilon \end{bmatrix}$$

Eigenvalues of A :

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\varepsilon - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + \varepsilon\lambda + 1 = 0$$

$$\lambda = \frac{-\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}$$

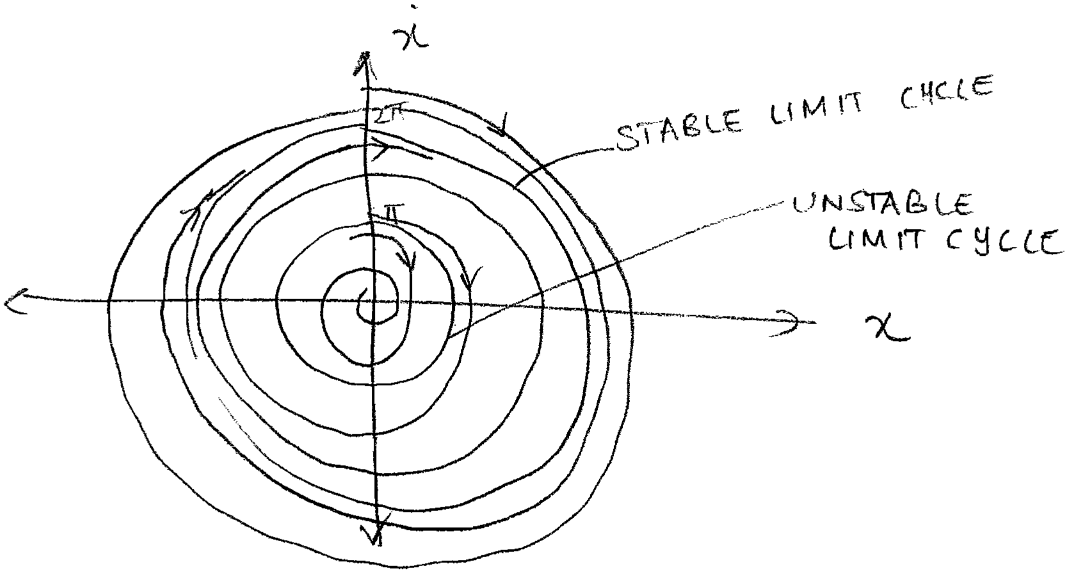
For $0 < \varepsilon < 1$, $\lambda$ is complex with negative real part $\therefore$ stable spiral

(c) For $0 < \dot{x} < \pi$, damping is positive and amplitude decays

$\pi < \dot{x} < 2\pi$, damping is negative and amplitude grows

$2\pi < \dot{x} < 3\pi$, damping is positive and amplitude decays

and so there is a periodicity of $2\pi$ in the damping. This suggests an infinite set of limit cycles in the phase plane.
LIMIT CYCLE is stable when all neighbourhood trajectories converge to it.
There is now the possibility of an infinite set of limit cycles, the first is unstable, the second stable, the third unstable and so on.




(d) If $\varepsilon$ is negative, $|\varepsilon| < 1$, the structure of phase plane remains the same but stability of limit cycles is reversed.