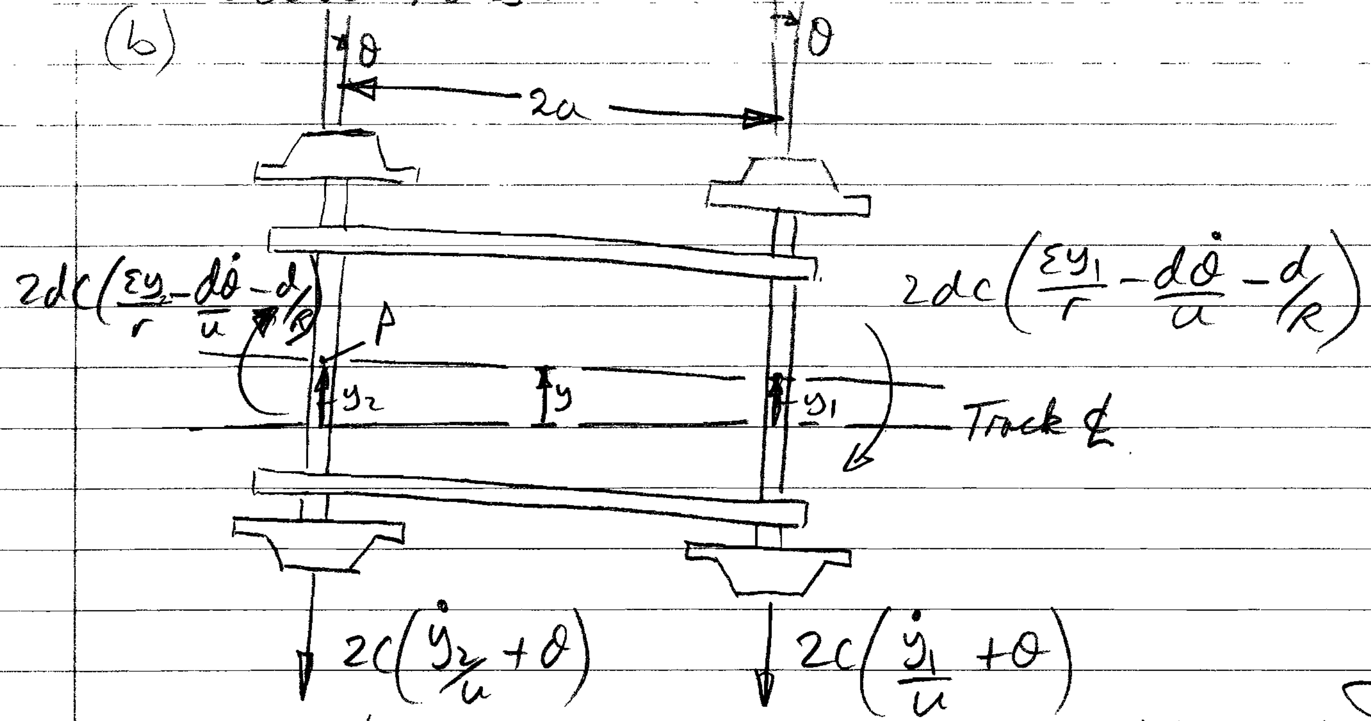
SOLUTIONS

1. (a) For derivation of the force & moment, see lecture notes

(b)



$$\sum M_P: \quad \cancel{2}d\cancel{c}\left(\frac{\varepsilon y_1}{r} - \frac{2d\dot\theta}{u} - \frac{2d}{R} + \frac{\varepsilon y_2}{r}\right) + (2a)\cancel{2c}\left(\frac{\dot y_1}{u} + \theta\right) = 0 \qquad (1)$$

Now $y_1 = y - a\theta$ & $y_2 = y + a\theta$ ——— (2)

$$\sum F_y = 0: \quad 2c\left(\frac{\dot y_1 + \dot y_2}{u} + 2\theta\right) = 0$$

$$\Rightarrow \quad \theta = -\left(\frac{\dot y_1 + \dot y_2}{2u}\right) = -\dot y / u \qquad (3)$$

$$\& \quad \dot\theta = -\ddot y / u \qquad (4)$$

Combining (1)–(4) gives a forced, undamped oscillator:

$$\ddot y + \underbrace{\left(\frac{\varepsilon}{dr}\,\frac{u^2}{(1 + a^2/d^2)}\right)}_{\omega_n^2} y = \left(\frac{u^2}{1 + a^2/d^2}\right)\frac{1}{R} \qquad (5)$$

Kinematic hunting wavelength $\lambda = \dfrac{2\pi u}{\omega_n} = \underbrace{2\pi\sqrt{\dfrac{dr}{\varepsilon}}}_{\lambda_{\text{free wheelset}}}\sqrt{1 + \dfrac{a^2}{d^2}}$

(c) If $z = \Delta \sin \frac{2\pi x}{L}$

$$\Rightarrow \frac{1}{R} \simeq \frac{d^2z}{dx^2} = -\frac{4\pi^2}{L^2}\Delta \sin \frac{2\pi x}{L}$$

Now put $x = Ut \Rightarrow \frac{1}{R} = -\frac{4\pi^2}{L^2}\Delta \sin \frac{2\pi U t}{L}$

& $\omega = \frac{2\pi U}{L}$ $\therefore$ $\frac{4\pi^2}{L^2} = \frac{\omega^2}{U^2}$

$$\text{So } \frac{1}{R} = -\frac{\omega^2}{U^2}\Delta \sin \omega t$$

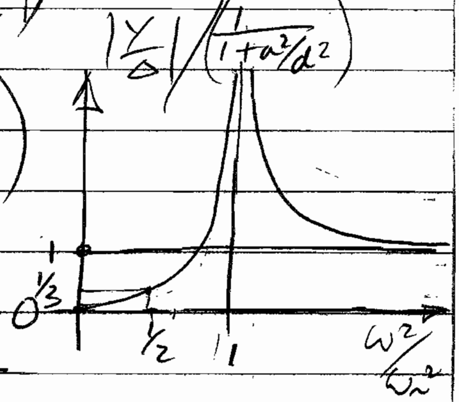
Let $y = Y e^{i\omega t}$

$$\Rightarrow (-\omega^2 + \omega_n^2)\, Y = -\left(\frac{\cancel{U^2}}{1 + a^2/d^2}\right)\frac{\omega^2}{\cancel{U^2}}\Delta$$

$$\therefore \frac{Y}{\Delta} = \frac{-1}{1 + a^2/d^2}\,\frac{\omega^2}{\omega_n^2 - \omega^2}$$

$$\left|\frac{Y}{\Delta}\right| = \frac{1}{(1 + a^2/d^2)}\left(\frac{\omega^2/\omega_n^2}{|1 - \omega^2/\omega_n^2|}\right)$$

(@ $\omega/\omega_n = 0$, $Y/\Delta = 0 \Rightarrow$ straight line motion with zero y ✓)



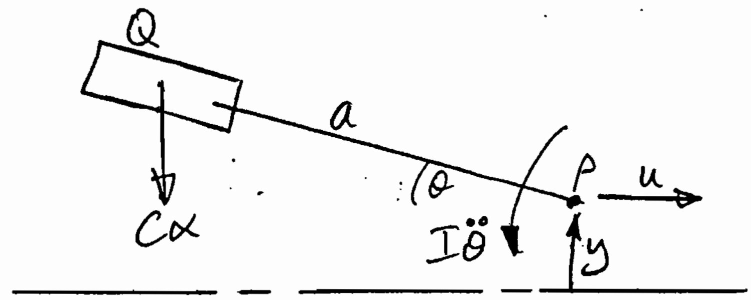
for $\lambda = \lambda_{\text{hunting}} \times 2$

$\Rightarrow \omega = \omega_n/2$

$$\Rightarrow \frac{Y}{\Delta} = \frac{1}{1 + a^2/d^2}\left(\frac{1/4}{1 - 1/4}\right) = \underline{\underline{\frac{1}{3(1 + a^2/d^2)}}}$$

$\omega/\omega_n \to \infty$, $\frac{Y}{\Delta} \to \frac{1}{1 + a^2/d^2}$

2.



Transverse velocity of Q is: $\dot{y} + a\dot{\theta}$

Lateral creep is: $\alpha = \dfrac{\dot{y} + a\dot{\theta}}{u} + \theta$ —— ①

$\circlearrowleft^{+} \Sigma M_P$: $\quad I\ddot{\theta} + aC\alpha = 0$ —— ②

using ①: $\quad I\ddot{\theta} + aC\left(\dfrac{\dot{y} + a\dot{\theta}}{u} + \theta\right) = 0$

$$\frac{I}{aC}\ddot{\theta} + \frac{a}{u}\dot{\theta} + \theta = -\dot{y}/u$$

$$\Rightarrow \quad \frac{\ddot{\theta}}{\omega_n^2} + \frac{2\zeta}{\omega_n}\dot{\theta} + \theta = -\frac{V}{u} \qquad \text{—— ③}$$

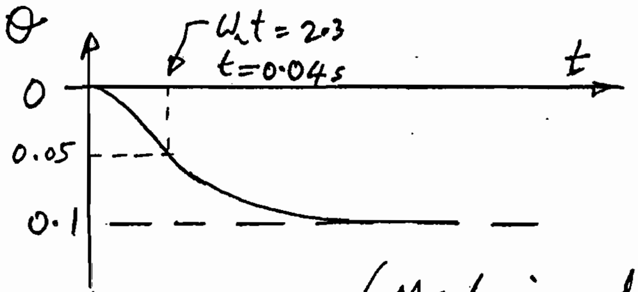
with $\omega_n^2 = \dfrac{aC}{I}$, $\quad \dfrac{2\zeta}{\omega_n} = \dfrac{a}{u} \quad \Rightarrow \zeta = \dfrac{a}{2u}\sqrt{\dfrac{aC}{I}}$

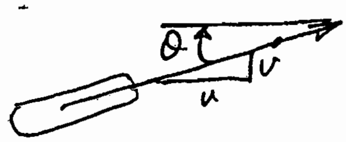
for
$a = 0.05\,m$
$I = 5 \times 10^{-4}\ kg\,m^2$
$V = 0.1\ m/s$
$u = 1\ m/s$
$C = 40\ N/rad$

$\Big\}$ $\omega_n = 63$ rad/s $= 10$ Hz

$\zeta = 1.6$

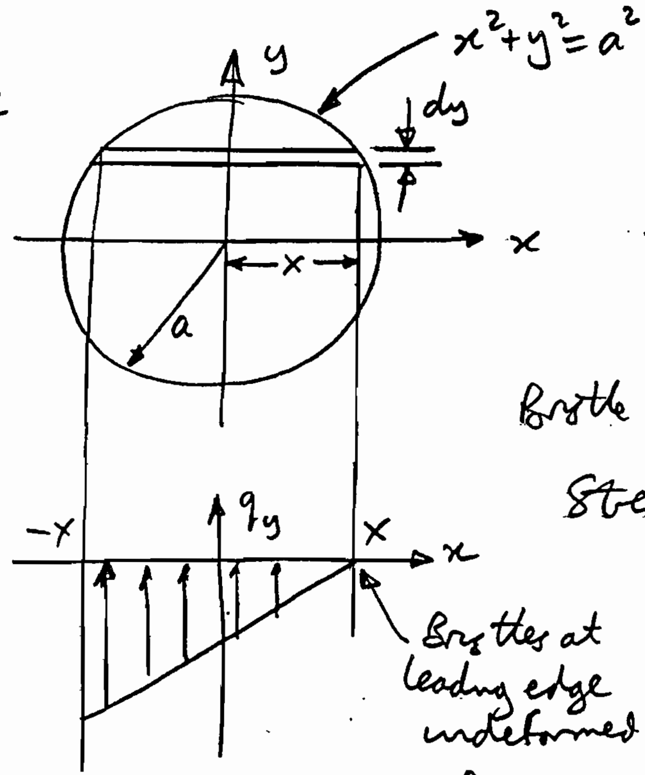
$V/u = 0.1$

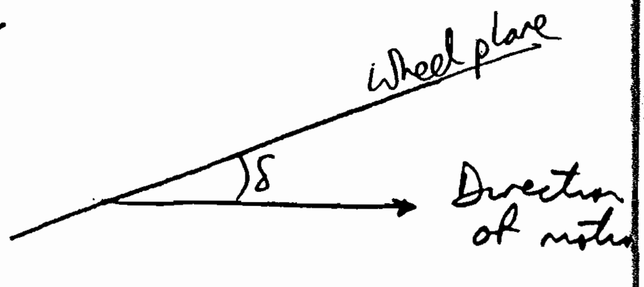




(Mechanics data book)

2b





Bristle Displacement: $q_y(x,y) = (X-x)\delta$

Stress: $\sigma_y(x,y) = K(X-x)\delta$

$\sigma_x(x,y) = 0$

Lateral force $\displaystyle Y = \iint_A \sigma_y \, dA = \int_{-a}^{a} dy \underbrace{\int_{-X}^{X} K(X-x)\,\delta \cdot dx}_{\text{Lateral force on a narrow strip of length } 2X}$

$$= K\delta \int_{-a}^{a} \left[Xx - \tfrac{x^2}{2} \right]_{-X}^{X} dy$$

$$= K\delta \int_{-a}^{a} 2X^2 \, dy$$

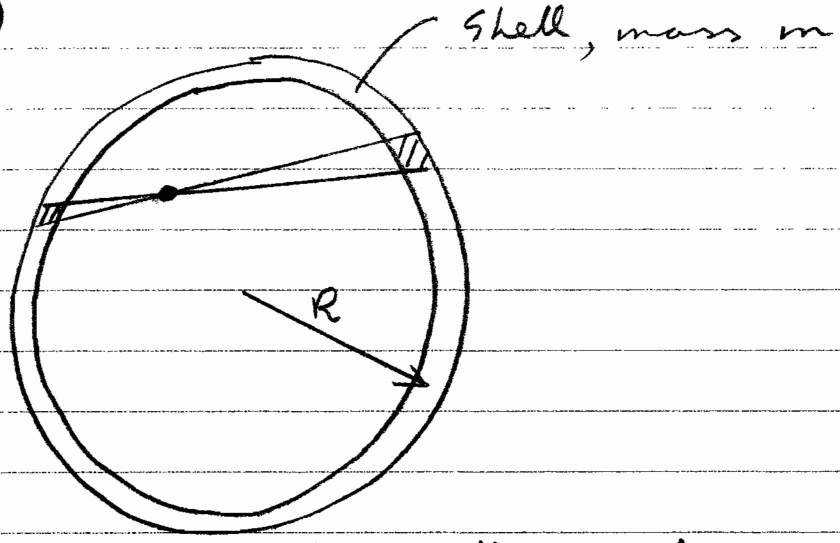
$$= 2K\delta \int_{-a}^{a} (a^2 - y^2)\, dy$$

$$= \underline{\underline{\tfrac{8}{3} a^3 K \delta}}$$

Assumptions:

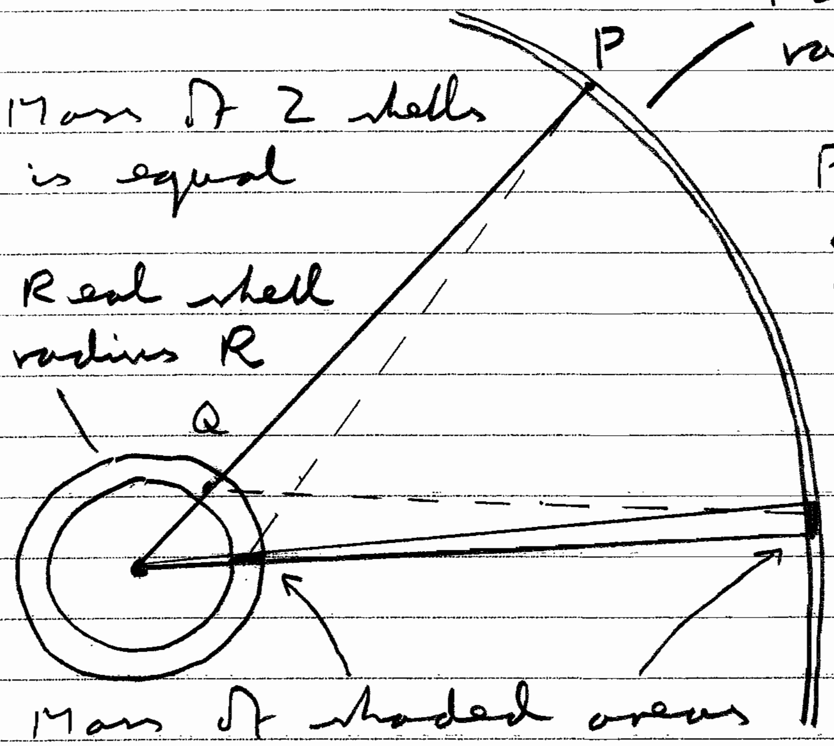
1. Brush model — Bristles are linear elastic, no inertia, not connected at tips
2. Small angles
3. No microslip

3 a)



Inside shell, pull is always equal and opposite, so potential is constant

= potential at centre = $\underline{mG/R}$



Potential at P due to real shell is same as potential at Q due to fictitious shell

$$= \frac{mG}{r}$$

3 (b) i) From section 5 of the data sheet,

$$\frac{\partial^2 u}{\partial \theta^2} + u = \frac{\mu u^2}{h^2 u^2} = \text{constant}$$

where $\mu = MG$

Solution is $u = \frac{MG}{h^2}(1 + e\cos\theta)$

Then $r = \dfrac{\frac{h^2}{MG}}{1 + e\cos\theta}$ — L in data sheet

ii) Now $E = \frac{1}{2}m(r\dot{\theta})^2 - \frac{mMG}{r}$ at periapsis

$$= \frac{m}{2r^2}h^2 - \frac{mMG}{r}$$

Also at periapsis, $r = \frac{h^2}{MG(1+e)}$

So $E = \frac{mM^2G^2(1+e)^2}{2h^2} - \frac{mM^2G^2(1+e)}{h^2}$

$$= \frac{mMG^2}{2h^2}(1 + 2e + e^2 - 2 - 2e)$$

$$= \frac{mMG^2}{2h^2}(e^2 - 1)$$

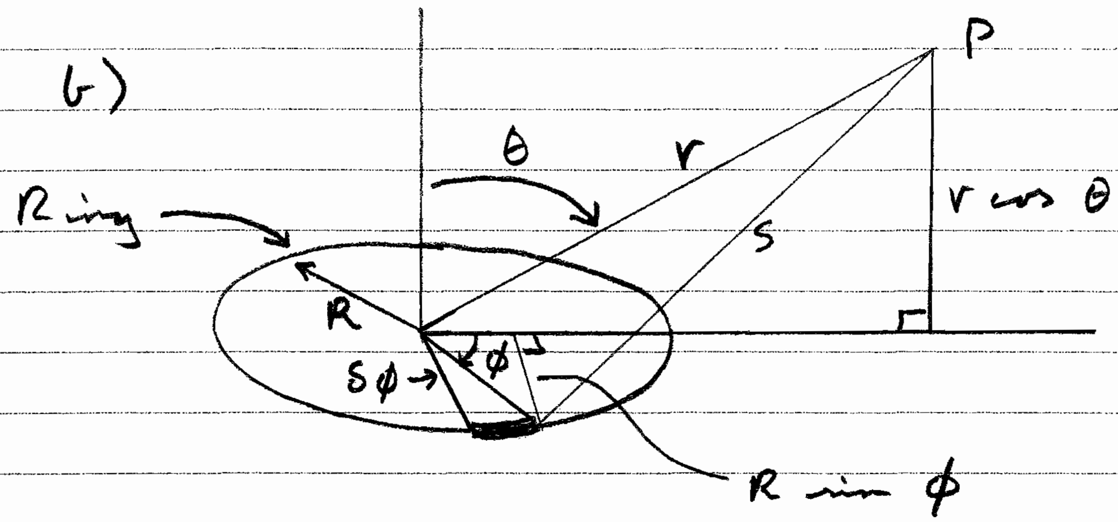
For a closed orbit, $0 \leq e \leq 1$

Minimum energy when $e = 0$,

i.e. a circular orbit.

4 a) J_2 is caused by the Earth's equatorial bulge, which results from the spin of the Earth about its Polar axis.

b)



$$\text{Mass / unit } l \text{ of ring} = \frac{m}{2\pi R}$$

$$\text{Potential at } P = \int_0^{2\pi} \frac{m}{2\pi R} \frac{RG}{s} \, d\phi$$

$$\text{But } s^2 = (R \sin\phi)^2 + (r\sin\theta - R\cos\phi)^2 + (r\cos\theta)^2$$

$$= R^2 \sin^2\phi + r^2\sin^2\theta - 2r\sin\theta R\cos\phi + R^2\cos^2\phi + r^2\cos^2\theta$$

$$= R^2 + r^2 - 2rR\sin\theta\cos\phi$$

$$\text{So potential} = \frac{mG}{2\pi} \int_0^{2\pi} \frac{1}{\sqrt{R^2 + r^2 - 2rR\sin\theta\cos\phi}} \, d\phi$$

c) Let $R/r = a$

$$U = \frac{mG}{2\pi} \int_0^{2\pi} r^{-1} (1 + a^2 - 2a\sin\theta\cos\phi)^{-1/2} \, d\phi$$

4c) cont.

If $r >> R$, then $a << 1$

$$\text{So } U \approx \frac{mG}{2\pi r}\int_0^{2\pi} 1 - \tfrac{1}{2}(a^2 - 2a\sin\theta\cos\phi) + \tfrac{3}{4}(a^2 - 2a\sin\theta\cos\phi)^2 \, d\phi$$

$$\approx \frac{mG}{2\pi r}\int_0^{2\pi} 1 - \frac{a^2}{2} + a\sin\theta\cos\phi + 3a^2\sin^2\theta\cos^2\phi \, d\phi$$

- ignoring higher terms.

$$= \frac{mG}{2\pi r}\int_0^{2\pi} 1 - \frac{a^2}{2} + ar\sin\theta\cos\phi + \tfrac{3}{2}a^2\sin^2\theta(1+\cos 2\phi)\, d\phi$$

$$= \frac{mG}{2\pi r}\left[\phi - \frac{a^2\phi}{2} + a\sin\theta\sin\phi + \tfrac{3}{4}a^2\sin^2\theta\left(\phi + \frac{\sin 2\phi}{2}\right)\right]_{\phi=0}^{\phi=2\pi}$$

$$= \frac{mG}{r}\left(1 - \frac{a^2}{2} + \tfrac{3}{4}a^2\sin^2\theta\right)$$

d) From this, we can write

$$U = \frac{mG}{r}\left(1 - \frac{a^2}{4}\left(2 - 3(1-\cos^2\theta)\right)\right)$$

$$= \frac{mG}{r}\left(1 - \frac{R^2}{r^2}\frac{(3\cos^2\theta - 1)}{2}\right) = \frac{mG}{r}\left(1 - \frac{R^2}{2r^2}P_2(\cos\theta)\right)$$

Adding a spherical mass M gives

$$U = \frac{(m+M)G}{r} - \frac{mG}{2r}\left(\frac{R^2}{r^2}\right)P_2(\cos\theta)$$

From data sheet; $U = \dfrac{(M+m)G}{r} - \dfrac{(M+m)G}{r}J_2\dfrac{R^2}{r^2}P_2(\cos\theta)$

$$\text{So } (M+m)J_2 = \frac{m}{2}$$

$$\text{i.e. } \left(\frac{m}{M+m}\right) = 2J_2 = 2\times 10^{-3}$$