1(a)(i) $e_{inm}e_{inm}$

$$\Rightarrow \delta_{nn}\delta_{mm} - \delta_{nm}\delta_{mn}$$

$$\Rightarrow \delta_{11}\delta_{11} + \delta_{11}\delta_{22} + \delta_{11}\delta_{33} + \delta_{22}\delta_{11} + \delta_{22}\delta_{22} + \delta_{22}\delta_{33} + \delta_{33}\delta_{11} + \delta_{33}\delta_{22} + \delta_{33}\delta_{33}$$

$$- \delta_{11}\delta_{11} - \delta_{22}\delta_{22} - \delta_{33}\delta_{33}$$

$$= 9 - 3 = 6$$

(ii) $e_{1jk}\delta_{3j}v_k$

$$\Rightarrow e_{13k}v_k$$

$$= e_{132}v_2 = -v_2$$

$e_{inm}e_{inq}$

$$\Rightarrow \delta_{nn}\delta_{mq} - \delta_{nq}\delta_{mn}$$

$$= 3\delta_{mq} - \delta_{nq}\delta_{mn}$$

$$= 3\{\delta_{11} + \delta_{22} + \delta_{33}\} - \delta_{11}\delta_{11} - \delta_{22}\delta_{22} - \delta_{33}\delta_{33}$$

$$= 9 - 3 = 6$$

(iii) $e_{ijk}e_{lmk}v_{m,jl}$

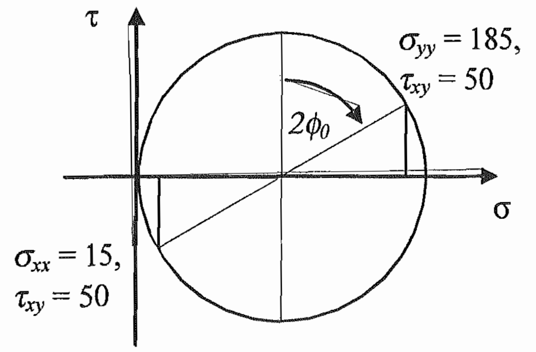
$$\Rightarrow e_{kij}e_{klm}v_{m,jl}$$

$$\Rightarrow (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})v_{m,jl}$$

$$\Rightarrow \delta_{il}\delta_{jm}v_{m,jl} - \delta_{im}\delta_{jl}v_{m,jl}$$

$$\Rightarrow v_{j,ji} - v_{i,jj}$$

1(b) At the origin, representing the state of stress on a Mohr's Circle gives,



So, at the origin the stress is equivalently described by $P_o = 100$MPa, $k = 100$MPa, $\phi_o = \pi/6$. k is a material constant.

Hencky's equations are $P\text{-}2k\phi = constant$ for α-characteristics, and $P+2k\phi = constant$ for β-characteristics. All values of ϕ can be estimated from the diagram.

Tracking along α-characteristic to point B, at B $\phi_B \cong \pi/12$, so $P_0 - 2k\phi_0 = P_B - 2k\phi_B$ and thus $P_B = P_0 - 2k(\phi_0 - \phi_B) = 100 - 200\left(\frac{\pi}{6} - \frac{\pi}{12}\right) = 48MPa.$

Now tracking along the β-characteristic to point A, where $\phi_A \cong \pi/4$.

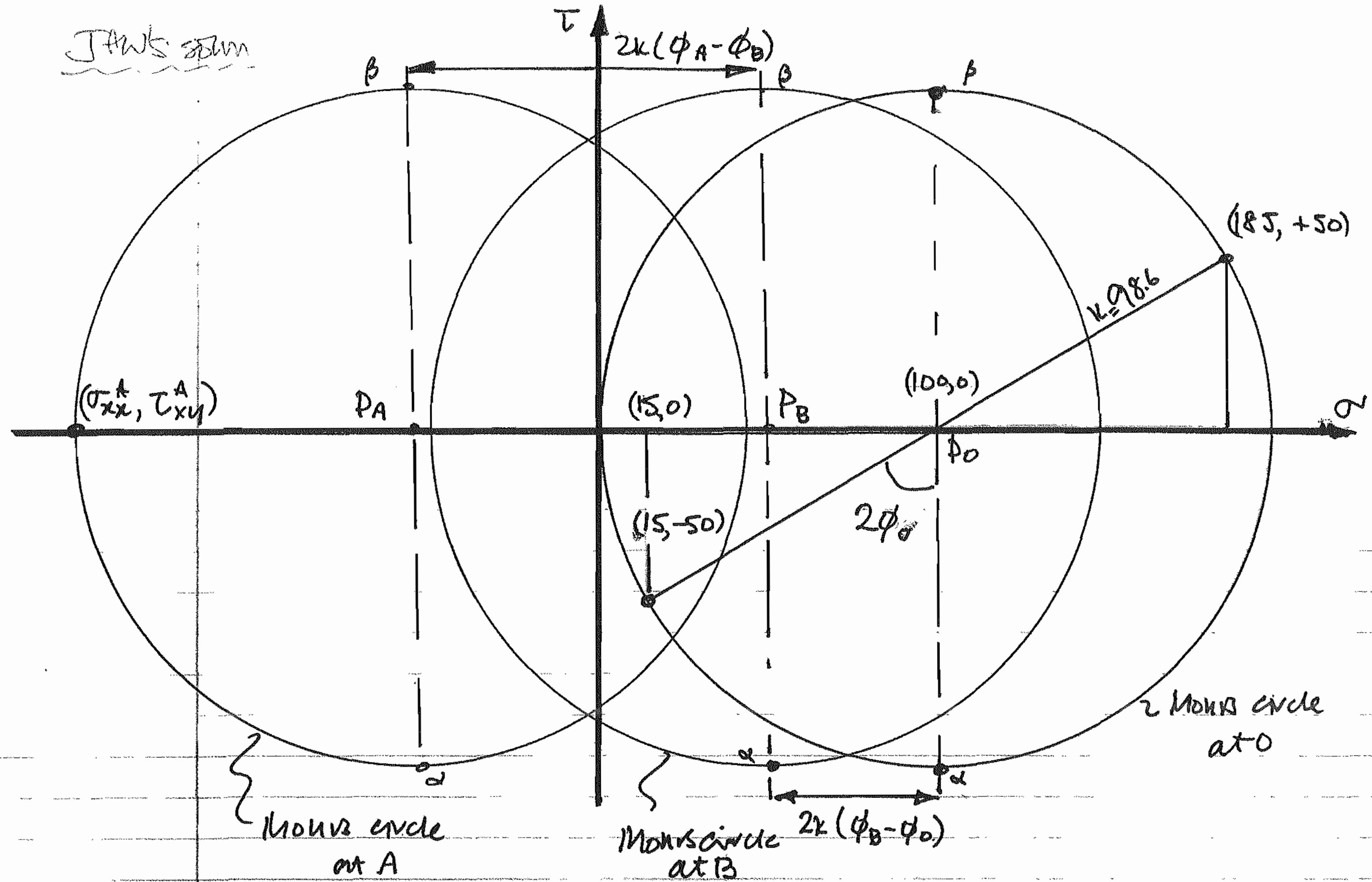
$P_B + 2k\phi_B = P_A + 2k\phi_A$ so $P_A = P_B + 2k(\phi_B - \phi_A) = 48 + 200\left(\frac{\pi}{12} - \frac{\pi}{4}\right) = -57MPa$.

Finally, converting back into the original co-ordinates,

$$\begin{cases} \sigma_{xx}^A = P_A - k\sin 2\phi_A = -157MPa \\ \sigma_{yy}^A = P_A + k\sin 2\phi_A = 43MPa \\ t_{xy}^A = k\cos 2\phi_A = 0 \end{cases}$$

Mohrs circle diagrams on following page

JAWS soln



By measurement from Fig. 1

$\phi_0 = 30^\circ$; $\phi_B = 15^\circ$; $\phi_A = 45^\circ$

Mohr's circle at O; $p_0 = 100\,\text{MPa}$; $k = \sqrt{85^2+50^2} = 98.6\,\text{MPa}$

[By calculation $\phi_0 = \frac{1}{2}\left(90^\circ - \tan^{-1}(50/85)\right) = 29.8^\circ$]

Along α-line; O → B

$$p_0 - 2k\phi_0 = p_B - 2k\phi_B$$

$$\therefore p_B = p_0 - 2k(\phi_0 - \phi_B)$$

$$\therefore p_B = 100 - 2\times 98.6\,(30-15)\pi/180^\circ$$

$$\Rightarrow 48.4\ \text{MPa}$$

Along β-line B → A

$$p_B + 2k\phi_B = p_A + 2k\phi_A$$

$$p_A = p_B + 2k(\phi_B - \phi_A)$$

$$= 48.4 + 2\times 98.6\,(15-45)\pi/180^\circ$$

$$\Rightarrow -54.8\ \text{MPa}$$

Thus

$$\sigma_{xx}^A = p_A - k = -54.8 - 98.6 = -153\,\text{MPa}$$

$$\sigma_{yy}^A = p_A + k = -54.8 + 98.6 = 44\,\text{MPa}$$

$$\tau_{xy}^A = 0$$

Taking other values of ϕ_0, ϕ_B, ϕ_A will give slightly different values of σ_{xx}^A, σ_{yy}^A, τ_{xy}^A.

2

(a) The stress tensor is $\sigma_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & \sigma_{zz} \end{bmatrix}$ so the deviatoric stress tensor is

$$\sigma'_{ij} = \frac{p+\sigma_{zz}}{3}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

The von Mises equivalent stress is thus $\bar{\sigma} = \sqrt{\tfrac{3}{2}\sigma'_{ij}\sigma'_{ij}} = \dfrac{p+\sigma_{zz}}{3}\sqrt{\tfrac{3}{2}}(1+1+4) = p+\sigma_{zz}$

(b) If $\sigma_{zz} = \alpha t$ then yield will first occur at t_1 where $p+\alpha t_1 = C(\varepsilon_0)^n$.

By the Levy-Mises flow rule, $\dot{\varepsilon}_{xx} = \dot{\varepsilon}_{yy} = -\dot{\lambda}\dfrac{p+\sigma_{zz}}{3}, \dot{\varepsilon}_{zz} = 2\dot{\lambda}\dfrac{p+\sigma_{zz}}{3}$. Thus $\dot{\varepsilon}_{xx} = \dot{\varepsilon}_{yy} = -\tfrac{1}{2}\dot{\varepsilon}_{zz}$ so the effective strain rate is $\dot{\bar{\varepsilon}} = \sqrt{\tfrac{2}{3}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}} = \dot{\varepsilon}_{zz}\sqrt{\tfrac{2}{3}\left(\tfrac{1}{4}+\tfrac{1}{4}+1\right)} = \dot{\varepsilon}_{zz}$

Taking the time derivative of the Swift law, $\dot{\bar{\sigma}} = \dfrac{n\dot{\bar{\varepsilon}}\bar{\sigma}}{\varepsilon_0+\bar{\varepsilon}}$, so rearranging and using the formula for the equivalent stress (and its time derivative) from (a),

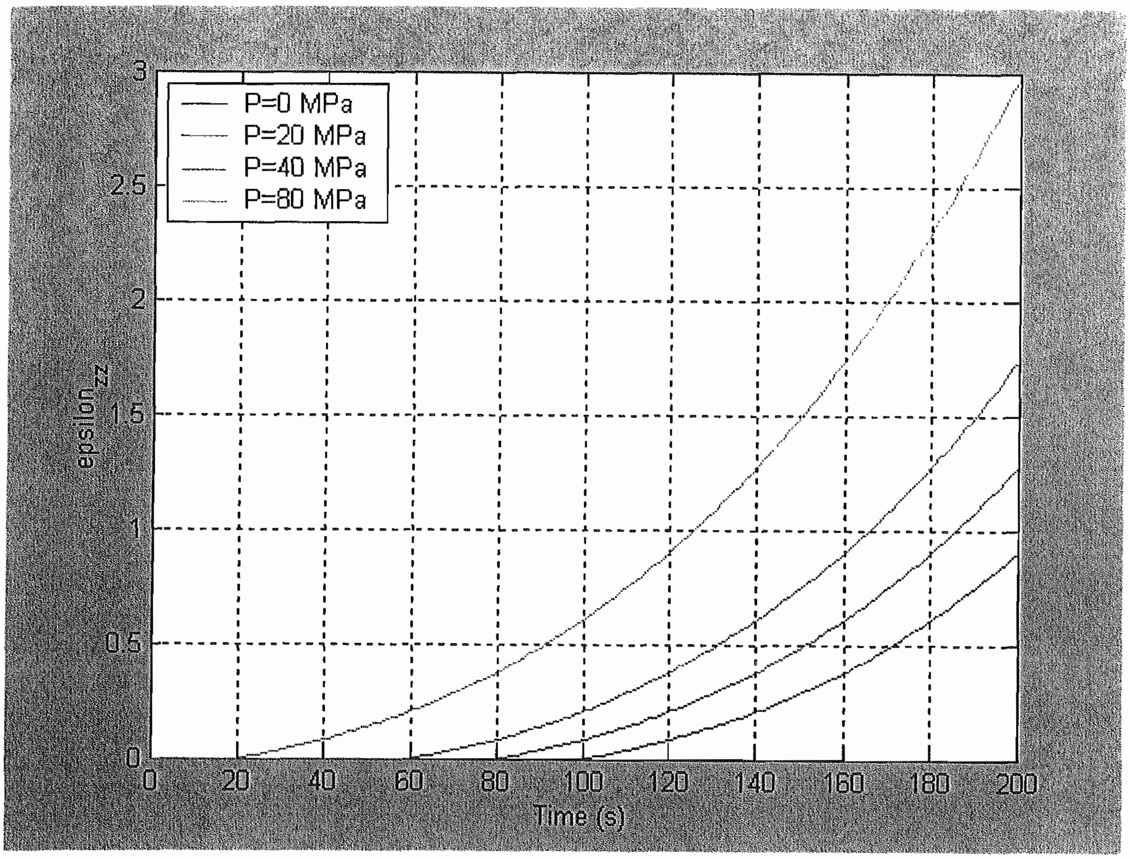
$$\dot{\bar{\sigma}}(\varepsilon_0+\bar{\varepsilon}) = n\dot{\bar{\varepsilon}}\bar{\sigma} \Rightarrow \alpha(\varepsilon_0+\bar{\varepsilon}) = n(p+\alpha t)\dot{\bar{\varepsilon}}.$$

Thus, $\displaystyle\int_{t_1}^{t}\frac{\alpha}{(p+\alpha t)}dt = \int_0^{\bar{\varepsilon}}\frac{n}{(\varepsilon_0+\bar{\varepsilon})}d\bar{\varepsilon}$, so $[\ln(p+\alpha t)]_{t_1}^{t} = [n\ln(\varepsilon_0+\bar{\varepsilon})]_0^{\bar{\varepsilon}}$

Finally, $\ln\left(\dfrac{p+\alpha t}{p+\alpha t_1}\right) = n\ln\left(\dfrac{\varepsilon_0+\bar{\varepsilon}}{\varepsilon_0}\right)$ so $\varepsilon_{zz} = \bar{\varepsilon} = \varepsilon_0\left[\left(\dfrac{p+\alpha t}{p+\alpha t_1}\right)^{\frac{1}{n}} - 1\right]$ $for\ t > t_1$

otherwise zero.

(c) The graph below is plotted exactly in Matlab, over the range 0≤t≤200 s, with the values α=1MPa/s, C=200MPa, n=0.3, $\varepsilon_0 = 0.1$, and for P=[0 20 40 80] MPa.



Increasing pressure leads to earlier yield, and greater extension for a given stress along the wire.

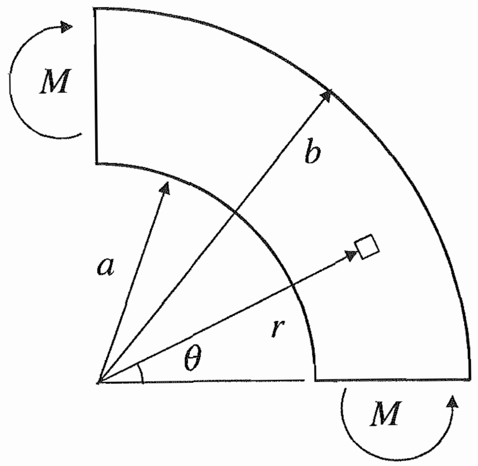
Examiners' Comments

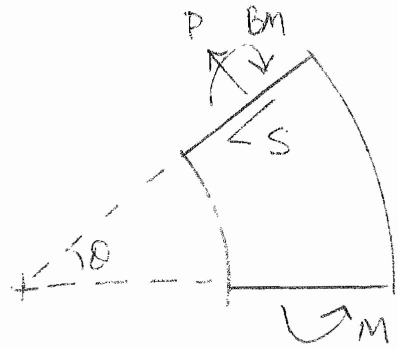
Q1 Part (a) was well done. Part (b) numerical answers depend on measuring rotation of α & β lines so precision not expected: several good solutions.

Q2 Errors in part (b) tended to be more algebraic than conceptual: necessary integration not well done. Curve sketching likewise.

Q3 Similar to a beam problem on an Examples Paper. Parts (a) & (b) well done. Part (c) given in this form to obviate need for excessive algebra – several complete solutions.

3 (a)





Equbm $\begin{cases} P\cos\theta = S\sin\theta \\ P\sin\theta + S\cos\theta = 0 \\ Pr - M + BM = 0 \end{cases}$

requires $\underline{P = S = 0 \text{ and } BM = M}$

Stresses are independent of θ

$$\therefore \quad \phi = Ar^2 + Br^2\ln r + C\ln r + D\theta$$

(b) from Table I

$$\begin{cases} \sigma_{rr} = 2A + B(2\ln r + 1) + C/r^2 \\ \sigma_{\theta\theta} = 2A + B(2\ln r + 3) - C/r^2 \\ \sigma_{r\theta} = D/r^2 \end{cases}$$

Four boundary conditions for 4 unknowns A, B, C & D

(c) now $\sigma_{rr} = 0$ at $r = a$ & b

$$\therefore \quad 2A + B + 2B\ln a + C/a^2 = 0$$

i.e. $\underline{(2A+B)a^2 + 2Ba^2\ln a + C = 0}$ — ①

and $\underline{(2A+B)b^2 + 2Bb^2\ln b + C = 0}$ — ②

$\sigma_{r\theta} = 0$ so clearly $\underline{D = 0}$ — ③

But also $\underline{\int_a^b r\,\sigma_{\theta\theta} = -M}$ — ④

⊛ If we note that $\sigma_{\theta\theta} = \dfrac{\partial^2\phi}{\partial r^2}$

then $$-M = \int_a^b r\frac{\partial^2\phi}{\partial r^2}\,dr$$

i.e. $$-M = \left[r\frac{\partial\phi}{\partial r}\right]_a^b - \int_a^b \frac{\partial\phi}{\partial r}\,dr$$

ie $\quad -M = \left[r\dfrac{\partial\phi}{\partial r}\right]_a^b - \left[\phi\right]_a^b$

But $\quad \tau_{rr} = \dfrac{1}{r}\dfrac{\partial\phi}{\partial r} \quad$ so $\quad r\dfrac{\partial\phi}{\partial r} = r^2\tau_{rr}$

ie. $\quad -M = \left[r^2\tau_{rr}\right]_a^b - \left[\phi\right]_a^b$

But $\quad \tau_{rr} = 0 \quad$ at $r = a$ & $b \quad$ so $\quad \left[r^2\tau_{rr}\right]_a^b \Rightarrow 0$

$$\therefore \quad M = \phi_b - \phi_a$$

$$M = Ab^2 + Bb^2\ln b + C\ln b - Aa^2 - Ba^2\ln a - C\ln a$$

ie, $\quad M = A(b^2 - a^2) + C\ln b/a + B(b^2\ln b - a^2\ln a) \quad$ — ⑤

But from ① & ②

$$2Bb^2\ln b + (2A+B)b^2 = 2Ba^2\ln a + (2A+B)a^2$$

$$\therefore \quad 2B(b^2\ln b - a^2\ln a) = (2A+B)(a^2 - b^2) \quad = -2A(b^2-a^2) - B(b^2-a^2)$$

$\therefore$ from ⑤ $\quad M = A(b^2 - a^2) + C\ln b/a + \tfrac{1}{2}(2A+B)(a^2-b^2)$

$$= A(b^2-a^2) - A(b^2-a^2) - \frac{B}{2}(b^2-a^2) + C\ln b/a$$

ie $\quad M = -\dfrac{B}{2}(b^2-a^2) + C\ln b/a$

hence $\quad \dfrac{M}{B} = -\dfrac{(b^2-a^2)}{2} + \dfrac{C}{B}\ln b/a$

Now substitute from given expressions

$$\frac{M}{B} = \frac{-N}{2(b^2-a^2)} = -\frac{(b^2-a^2)}{2} + \frac{2a^2b^2\ln b/a \,.\, \ln b/a}{b^2-a^2}$$

$$\therefore \quad N = (b^2-a^2)^2 - 4a^2b^2(\ln b/a)^2$$

⊛ Aside if not spoiled

$$-M = \int_a^b r\sigma_{\theta\theta}\,dr$$

$$-M = \int_a^b 2Ar + 3Br + 2Br\ln r - \frac{C}{r}\,dr$$

$$-M = \left[\frac{(2A+3B)r^2}{2}\right]_a^b - C\left[\ln r\right]_a^b + 2B\int r\ln r\,dr$$

So consider $I = \int_a^b r\ln r\,dr$

$$= \left[\frac{r^2}{2}\ln r\right]_a^b - \int_a^b \frac{r^2}{2}\cdot\frac{1}{r}\,dr$$

$$\Rightarrow \quad = \frac{b^2}{2}\ln b - \frac{a^2}{2}\ln a - \left[\frac{r^2}{4}\right]_a^b$$

$$I = \frac{1}{2}(b^2\ln b - a^2\ln a) - \frac{1}{4}(b^2-a^2)$$

$$\therefore -M = \frac{2A+3B}{2}(b^2-a^2) - C\ln b/a + B(b^2\ln b - a^2\ln a) - \frac{B}{2}(b^2-a^2)$$

$$-M = A(b^2-a^2) + B(b^2-a^2) - C\ln b/a + B(b^2\ln b - a^2\ln a)$$

$$M = -(A+B)(b^2-a^2) + C\ln b/a - B(b^2\ln b - a^2\ln a)$$

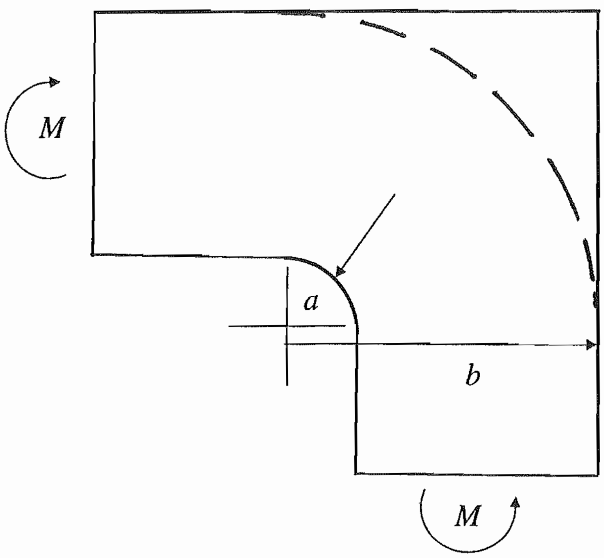
But also from ①

$$A(b^2-a^2) = -B(b^2\ln b \mp a^2\ln a) - \frac{B}{2}(b^2-a^2)$$

$$\therefore -M = -B(\cancel{b^2\ln b - a^2\ln a}) - \frac{B}{2}(b^2-a^2) + B(b^2-a^2) - C\ln b/a + B(\cancel{b^2\ln b - a^2\ln a})$$

ie. $M = -\frac{B}{2}(b^2-a^2) + C\ln(b/a)$ as before

(d)



Material outside dashed line cannot add significantly to much strength of component - so assume these stress free

letting, for convenience,

$n = b/a$

$$N = [(n^2-1)^2 - 4n^2(\ln n)^2]a^4$$

$$\sigma_{\theta\theta} = 2A + 2Br + 3B - C/r^2$$

when $r=a$ $\quad \sigma_{\theta\theta} = 2A + 2B\ln a + 3B - C/a^2$

$$= \frac{2M}{N}\{b^2 - a^2 + 2b^2\ln b - 2a^2\ln a\}$$

$$- \frac{4M}{N}(b^2-a^2)\ln a - \frac{6M}{N}(b^2-a^2)$$

$$+ \frac{4M}{N} b^2 \ln b/a$$ for component of unit thickness

ie. $\sigma_{\theta\theta} = \frac{4M}{N}\{n^2\ln n - n^2 + 1 + n^2\ln n\}a^2$

ie. $\sigma_{\theta\theta} = \frac{4(2n^2\ln n - n^2 + 1)}{(n^2-1)^2 - 4n^2(\ln n)^2} \cdot \frac{M}{a^2 t}$

Simple beam theory

$\frac{M}{I} = \frac{\sigma}{y}$ $\qquad I = \frac{(b-a)^3}{12}$ $\qquad y_{max} = \frac{b-a}{2}$

$\therefore \sigma = M \frac{b-a}{2} \frac{12}{(b-a)^3 t} = \frac{6M}{(n-1)^2 a^2 t}$

JRW
Jan 09.