Q1: Consider the mode shape given:
a)
$$\bar{U}_n = 1 - \cos\frac{n\pi x}{2L}$$

$$M_{eq} = \int_0^L m\,\bar{U}^2\,dx = \int_0^L m\left(1-\cos\frac{n\pi x}{2L}\right)^2 dx$$

$$= mL\left[\tfrac{3}{2} - \frac{4}{n\pi}\sin\frac{n\pi}{2}\right]$$

$$K_{eq} = \int_0^L EI\left(\frac{d^2\bar{U}}{dx^2}\right)^2 dx$$

$$\frac{d^2\bar{U}_n}{dx^2} = \left(\frac{n\pi}{2L}\right)^2 \cos\frac{n\pi x}{2L}$$

$$K_{eq} = EI\left(\frac{n\pi}{2L}\right)^4 \int_0^L \cos^2\frac{n\pi x}{2L}\,dx = EI\left(\frac{n\pi}{2L}\right)^4 \frac{L}{2}$$

Consider mode 1 ie $n=1$ $m = 200$ kg/m $EI = 8\times10^6$ Nm^2.
$L = 6$ m.

$$M_{eq} = 200\times 6\left[\frac{3}{2} - \frac{4}{\pi}\right] = 272.11 \text{ kg}$$

$$K_{eq} = 8\times10^6 \times \frac{\pi^4}{12^4} \times \frac{6}{2} = 0.1127\times10^6$$

$\therefore$ Natural frequency in mode 1 $= \sqrt{\dfrac{K_{eq}}{M_{eq}}} = 20.35$ rad/s or $\underline{\underline{3.24 \text{ Hz}}}$

Consider Mode 2 ie $n=2$

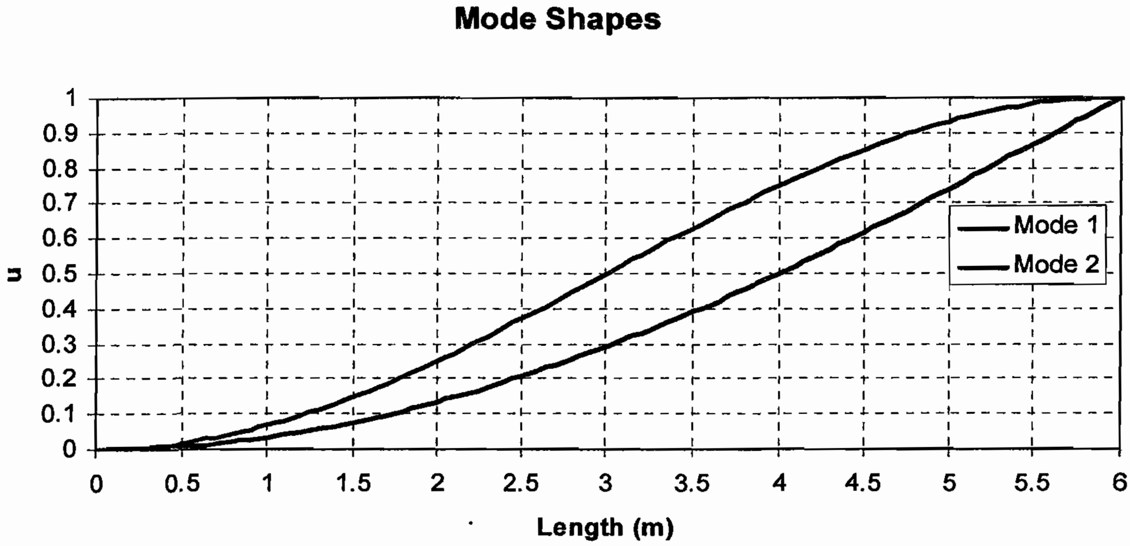
$$M_{eq} = 200\times 6\left[\frac{3}{2} - 0\right] = 1800 \text{ kg}$$

$$K_{eq} = 8\times10^6 \times \frac{6}{2} \times \left(\frac{\pi}{6}\right)^4 = 1.8039\times10^6$$

$\therefore$ Natural frequency in mode 2 $= \sqrt{\dfrac{K_{eq}}{M_{eq}}} = 31.65$ rad/s or $\underline{\underline{5.04 \text{ Hz}}}$.

[30%]

Q1: b)



Note: Mode 2 has been scaled down by a factor of 2 to make maximum ordinate as '1'.

[10%]

Q1c) Consider mode 1: $f_1 = 3.24$ Hz $\Rightarrow T_1 = 0.309$ s.

$t_d = 0.1$ Sec $\Rightarrow t_d/T_1 = \frac{0.1}{0.309} = 0.324$

From Data Sheet DAF = 1.65

$$\delta_{static} = \frac{F_{eq1}}{K_{eq1}} = \frac{10 \times 1000}{0.1127 \times 10^6} = 0.0887 \text{ m}.$$

$\therefore$ Max deflection in Mode 1 = DAF . δ_{static} = 0.0887×1.65

= 0.1464 m.

Consider Mode 2: $f_2 = 5.04$ Hz $\Rightarrow T_2 = 0.1985$ s.

$t_d/T_2 = \frac{0.1}{0.1985} = 0.504$

$\therefore$ DAF = 2

$$\delta_{static} = \frac{F_{eq2}}{K_{eq2}} = \frac{20 \times 1000}{1.8039 \times 10^6} = 0.0111 \text{ m}.$$

$\therefore$ Max deflection in mode 2 = DAF . δ_{static} = 0.0222 m.

Using SRSS method: Max tip deflection = $\sqrt{0.1464^2 + 0.0222^2} = 0.148$ m.

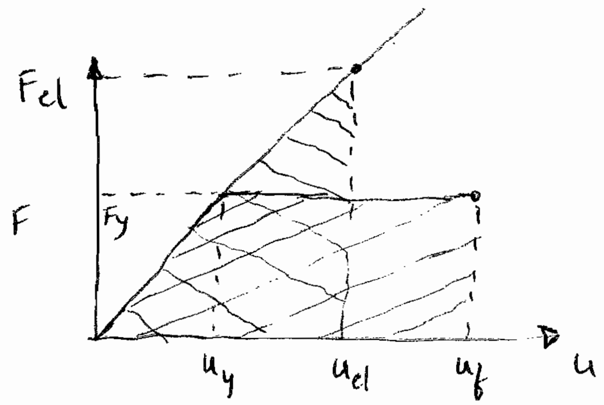
Note: Contribution of 2^{nd} mode is quite small $\simeq 150$ mm!

[30%]

Q 1 d) Ductility factor μ is defined as the ratio of displacement at failure u_f to the displacement at yield u_y.

$$\mu = \frac{u_f}{u_y}$$

Compare deformation energies in an elastic and elasto-plastic system



Elastic system : $\frac{1}{2} F_{el} \cdot u_{el}$ = Elasto plastic system = $F_y u_f - \frac{1}{2} F_y u_y$

$$\frac{1}{2} F_{el} u_{el} = F_y u_f - \frac{1}{2} F_y u_y$$

Divide by $F_y u_y$

$$\frac{1}{2} \frac{F_{el}}{F_y} \frac{u_{el}}{u_y} = \frac{u_f}{u_y} - \frac{1}{2}$$

But $\mu = \frac{u_f}{u_y}$, $\frac{F_{el}}{F_y} = \frac{u_{el}}{u_y}$

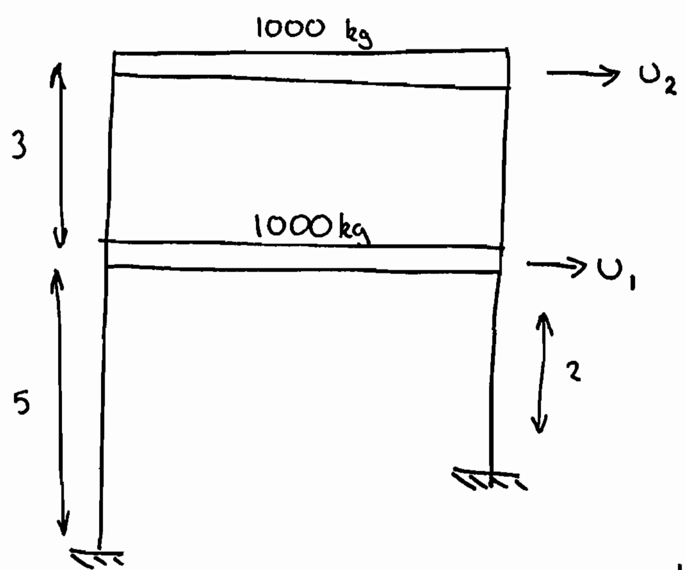
$$\Rightarrow \left(\frac{u_{el}}{u_y}\right)^2 = 2\mu - 1$$

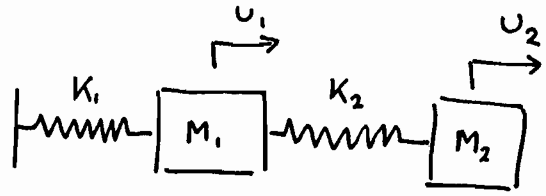
$$\Rightarrow \frac{F_{el}}{F_y} = \sqrt{2\mu - 1}$$

$$\Rightarrow F_y = \frac{F_{el}}{\sqrt{2\mu - 1}}$$ QED [30 %]

This method of comparing deformation energies is suitable for structures with intermediate freq range $2 < f_n < 8$ Hz. (not)

2 a)





$$K_1 = \frac{12EI}{2^3} + \frac{12EI}{5^3} = 15{\cdot}96 \text{ MN/m}$$

$$K_2 = 2 \times \frac{12EI}{3^3} = 8{\cdot}88 \text{ MN/m}$$

Equations of motion

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Put $u_1 = U_1 \sin\omega t$ $\quad$ $u_2 = U_2 \sin\omega t$

$\ddot{u}_1 = -U_1 \omega^2 \sin\omega t$ $\quad$ $\ddot{u}_2 = -U_2 \omega^2 \sin\omega t$

$$\begin{bmatrix} 24840 - \omega^2 & -8880 \\ -8880 & 8880 - \omega^2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(24840 - \omega^2)(8880 - \omega^2) - 8880^2 = 0$$

$$\omega^4 - 33720\,\omega^2 + 141724800 = 0$$

$$\omega^2 = \frac{33720 \pm \sqrt{570139200}}{2} = \begin{matrix} 4921 \\ \text{or} \\ 28799 \end{matrix}$$

$\omega_1 = 70.15$ rad/s 11.16 Hz

$\omega_2 = 169.7$ rad/s 27 Hz

For ω_1 : $19919\,u_1 - 8880\,u_2 = 0$

$$\frac{u_1}{u_2} = 0.446$$

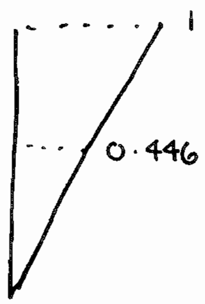
ω_2 : $-3959\,u_1 - 8880\,u_2 = 0$

$$\frac{u_1}{u_2} = -2.24$$

Mode shapes :

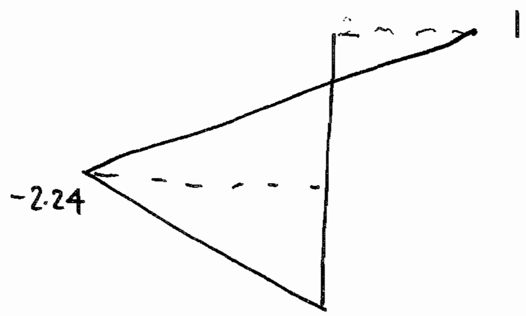
ω_1

$T_1 = 0.09s$



ω_2

$T_2 = 0.037s$



b) $M_{eq1} = 1000 + 1000 \times 0.446^2$

$= 1199 \text{ kg}$

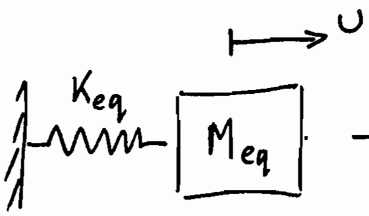
$M_{eq2} = 1000 + 1000 \times (-2.24)^2$

$= 6018 \text{ kg}$

$\frac{t_d}{T_1} = \frac{2 \times 10^{-2}}{0.09} = 0.22 \qquad DAF_1 = 0.6$

$\frac{t_d}{T_2} = \frac{2 \times 10^{-2}}{0.037} = 0.54 \qquad DAF_2 = 1.35$

Equivalent SDOF system



$\longrightarrow \quad F_{eq} = -\Gamma^1 M_{eq} \ddot{v}$

$K_{eq1} = 15960 \times 0.446^2 + 8880 \times 0.554^2$

$= 5900 \text{ kN/m}$

$K_{eq2} = 15960 \times (-2.24)^2 + 8880 \times 3.24^2$

$= 173{,}300 \text{ kN/m}$

$$\Gamma_1 = \frac{M_1 \bar{\upsilon}_1 + M_2 \bar{\upsilon}_2}{M_1 \bar{\upsilon}_1^2 + M_2 \bar{\upsilon}_2^2} = \frac{1000 \times 0.446 + 1000}{1000 \times 0.446^2 + 1000}$$

$$= 1.206 \quad \checkmark$$

$$\Gamma_2 = \frac{-2.24 + 1}{(-2.24)^2 + 1} = -0.206 \quad \checkmark$$

$$F_{eq_1} = -\Gamma_1 M_{eq_1} \ddot{v} = 1.206 \times 1199 \times 5 = -7230 \text{ N} \quad \checkmark$$

$$F_{eq_2} = -(-0.206) \times 6018 \times 5 = 6199 \text{ N} \quad \checkmark$$

$$\delta_{1\,dyn} = DAF_1 \times \frac{F_{eq_1}}{K_{eq_1}} = 0.6 \times \frac{-7230}{5900 \times 10^3} = -0.735 \text{ mm} \quad \checkmark$$

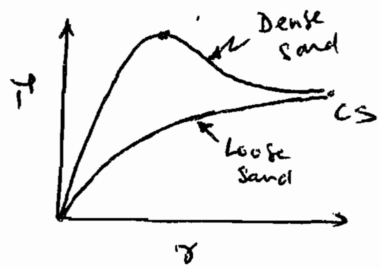
$$\delta_{2\,dyn} = \frac{1.35 \times 6199}{173{,}300 \times 10^3} = 0.048 \text{ mm} \quad \checkmark$$

Could use SRSS for superposition

$$\delta_{TOTAL} = \sqrt{0.048^2 + 0.735^2} = \underline{\underline{0.737}} \text{ mm} \quad \checkmark$$

Similar to mode 1 which dominates response. ✓

3 a)

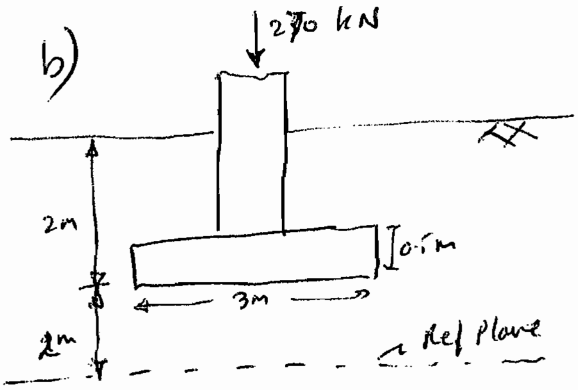


Loose saturated soils suffer liquefaction as the cyclic loading induced by earthquakes causes loose sand to densify. This tendency to suffer volumetric contraction is manifested as an increase in ex pore water pressure. If this excess pore water pressure equals the total stress then the loose sand suffer full liquefaction, as excess pp cannot be drained dury eq.

Dense sand, on the other hand, reach a peak strength and suffer volumetric dilation to reach the critical state. This causes a lowering of excess pore water causes the effective stress to increase dury the eq loading. This is why dense sands are thought not to suffer liquefaction dury earthquake loading. However, one should not rely on this increased effective stress as it will be lost once the pore water will migrate to equilibriate the pore pressures once the eq loading is finished.

[15%]

b)



Bearing pressure $= \frac{270}{3\times3} = 30$ kPa

Vertical stress on Ref plane

$\sigma_v = 30 + 2\times18.5 = 67$ kPa

Pore water pressure $u = 4\times10 = 40$ kPa

$\therefore \sigma_v' = \sigma_v - u = 27$ kPa.

$K_0 = \frac{\nu}{1-\nu} = \frac{0.3}{0.7} = 0.429 \quad \therefore p' = \left(\frac{1+2K_0}{3}\right)\sigma_v' = 16.71$ kPa

$\therefore G_{max} = 100\frac{[3-e]^2}{1+e}\sqrt{p'} = 100\frac{[3-0.85]^2}{1.85}\sqrt{\frac{16.71}{1000}} = 249.86\times0.129$

$= 32.31$ MPa

Horizontal Stiffness $K_{hx} = \frac{Gb}{2-\nu}\left[6.8\left(\frac{l}{b}\right)^{0.85} + 2.4\right]\left[1 + \left(0.33 + \frac{1.34}{1+l/b}\right)\left(\frac{e}{b}\right)^{0.8}\right]$

$2b = 3m$ $\quad \nu = 0.3 \quad l/b = 1 \quad \frac{e}{b} = \frac{2}{1.5} = 1.3\dot{3}$

$b = 1.5m$

$$\therefore K_{hx} = G \cdot \frac{1.5}{1.7} \times 9.2 \times (1 + 1.33^{0.8}) = 18.336\ G$$

$$K_{hx} = 32.31 \times 10^6 \times 18.336 = 592.45\ MN/m$$

Rotational Stiffness $K_{ry} = \frac{Gb^3}{1-\nu}\left[3.73\left(\frac{l}{b}\right)^{2.4} + 0.27\right]\left[1 + \frac{e}{b} + \frac{1.6}{0.35 + (l/b)^4}\left(\frac{e}{b}\right)^2\right]$

$$= G \times \frac{1.5^3}{0.7} \times 4 \times (1 + 1.3\dot{3} + 1.185 \times 1.3\dot{3}^2)$$

$$= 85.6349\ G = 2766.94\ MN \cdot m/rad$$

[30%]

3c) Horizontal Vibrations

Mass = 27000 kg.

$K_{hx} = 592.45\ MN/m$

$$\therefore \omega_h = \sqrt{\frac{K_{hx}}{M}} = 148.13\ rad/s = \underline{\underline{23.58\ Hz}}$$

Rocking Vibrations: (assuming foundation rocks about the base) & Ht of lumped mass above foundation = 2 + 2 = 4m

$$I = ma^2 = 27000 \times 4^2 = 432000\ kg\text{-}m^2$$

$$\therefore \omega_{rock} = \sqrt{\frac{K_{ry}}{I}} = \sqrt{\frac{2766.94 \times 10^6}{432000}} = 80.00\ rad/s = \underline{\underline{12.73\ Hz}}$$

[25%]

3 d) During strong earthquake, the rocking frequency reduces by 75%.

$\therefore f_{rock} = 0.25 \times 12.73 = \underline{3.184}$ Hz

$w_{rock} = 20$ rad/s.

$\therefore K_{ry} = I \cdot w_{rock}^2$

$= 432000 \times 20^2 = 172.93$ MN.m/rad

$\therefore \quad K_{ry} = G_{new} \times 85.6349$.

$\therefore \quad G_{new} = 2.02$ MPa

$K_{hx} = 18.336\, G = 37.028$ MN/m

$\therefore \quad w_{hx} = \sqrt{\dfrac{37.028}{27000}} = 37.03$ rad/s $= \underline{5.9}$ Hz

So the natural frequency in the horizontal direction also decreases by $\underline{75\%}$!

The soil-column system is vulnerable to rocking vibrations as the natural frequency has now dropped to $\underline{3.18}$ Hz, as most earthquakes have a frequency range of 1~5 Hz. It is less so to horizontal vibrations as the natural frequency is still 5.9 Hz > 1-5 Hz range. (but only marginally.). A more stronger earthquake can bring this further down. [30%]

Q4. a) i) High marks could be obtained by explaining the sequence from $S_{uu} \rightarrow S_{ff} \rightarrow S_{xx}$ via the aerodynamic and mechanical admittances as per the lectures, culminating in $\sigma_x^2 = \int S_{xx}(\omega)\, d\omega$

ii) Shortcomings include – the assumption that the structure does not change the statistics of the wind velocity spectrum; that the wind can be represented by a statistical distribution that may ignore physical phenomena such as "first-gust" in thunderstorms; etc.

iii) The main point concerns the fact that wind forces may be spatially de-correlated.

iv) Marks could be obtained by explaining, briefly, the way that the structural response is mode-generalised, leading to S_{xx} being a double summation over double integrals, as per the lecture notes.

b) Full marks could be obtained by proper explanation of the three techniques of increased stand-off, robustness and glazing design. In the latter case students should describe the various options and their shortcomings, with mention of the positive advantages of laminated glass with PVB interlayer, as per lectures.