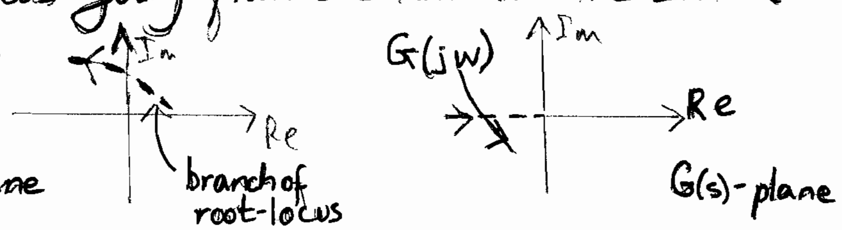
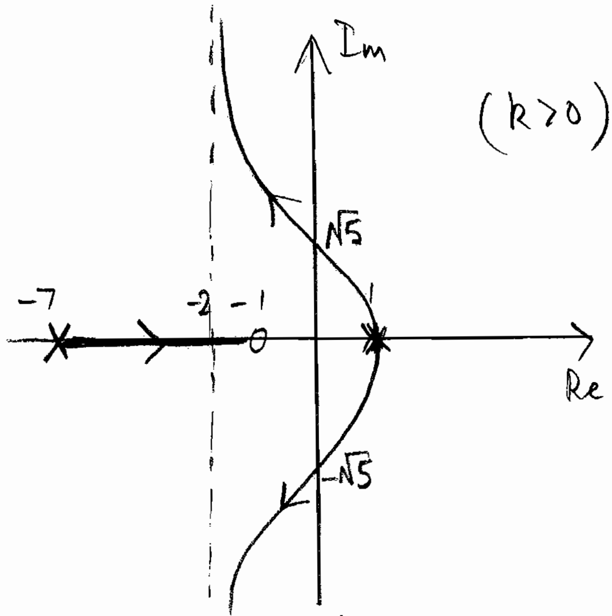
1(a)(i) Consider the mapping $s \mapsto G(s)$ which maps the "s-plane" to the "G(s)-plane". The Nyquist diagram is the image of the imaginary axis under this mapping, and lives in the G(s)-plane. The branches of the root-locus diagram map to the negative real axis under this mapping. Thus, the root-locus diagram lives in the s-plane and can be thought of as the "inverse image" of the negative real axis under the mapping.

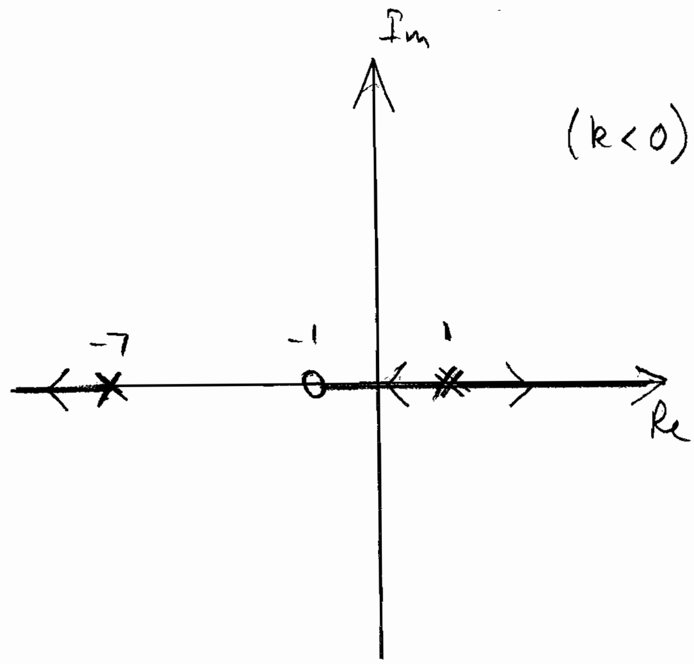
(ii) $G(s)$ is conformal at s_0 if it is analytic at s_0 and $\frac{d}{ds}G(s) \neq 0$ at s_0.
Conformal mappings preserve angle and their sense.

(iii) When k is small the number of closed-loop poles in the RHP is the same as the number of open-loop poles (poles of $G(s)$) in the RHP. As k is increased there is no change in the number of RHP closed-loop poles until a crossing of the negative real axis at $-1/k$ by the Nyquist diagram is encountered.
If the Nyquist diagram crosses from the upper to the lower half plane this corresponds to a branch of the root-locus going from the RHP into the LHP (and vice-versa).



1(b)(i)



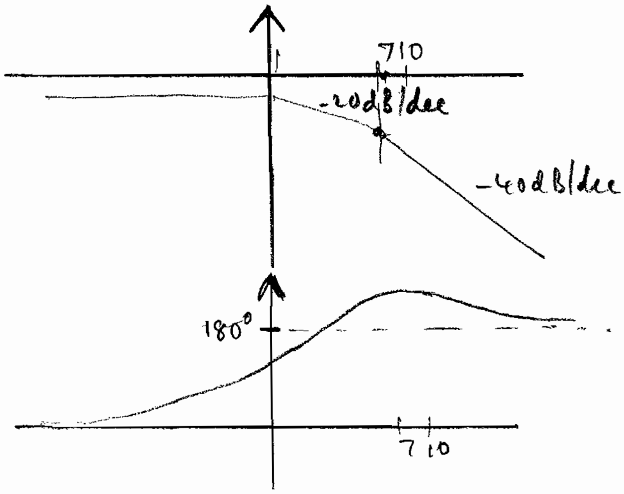


asymptote centre = $\frac{-7+2-(-1)}{2} = -2$

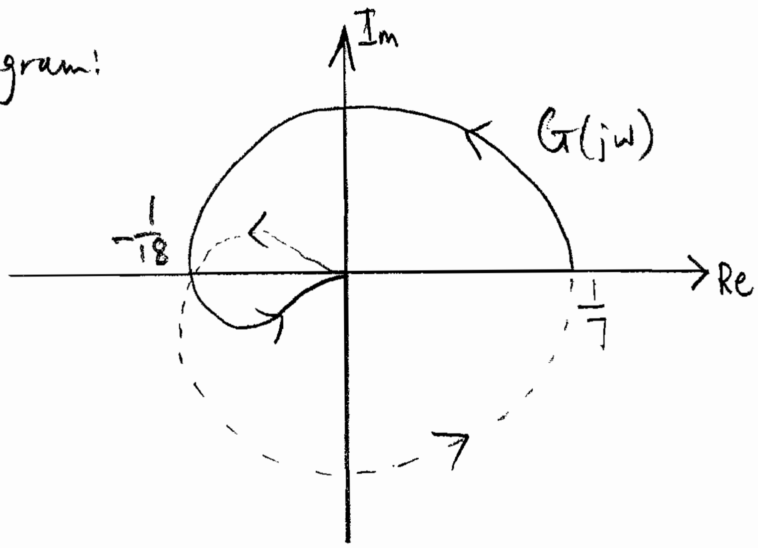
No need to solve for breakaway points. (Solving analytically will give $s=1$ and two complex locations.)

(ii)

Bode plot:



Nyquist diagram:



b(iii) Find w for which G(jw) is purely real.

$$G(jw) = \frac{jw+1}{(jw-1)^2(jw+7)} = \frac{(jw+1)^3(-jw+7)}{(w^2+1)^2(w^2+49)}$$

$$(s+1)^3(-s+7) = (s^3+3s^2+3s+1)(-s+7)$$
$$= (-s^4+18s^2+7) + (4s^3+20s)$$

$$G(jw) \text{ real} \iff 4(jw)^3 + 20(jw) = 0$$
$$\iff w = 0, \pm\sqrt{5}$$

$$G(0) = \frac{1}{7}$$

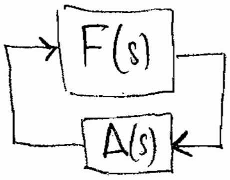
$$|G(\underbrace{j\sqrt{5}}_{w_0})|^2 = \left|\frac{1}{(jw_0-1)(jw_0+7)}\right|^2 = \frac{1}{(w_0^2+1)(w_0^2+49)} = \frac{1}{6\times 54}$$

$$\Rightarrow G(j\sqrt{5}) = -\frac{1}{18}$$ (minus sign from Bode diagram)

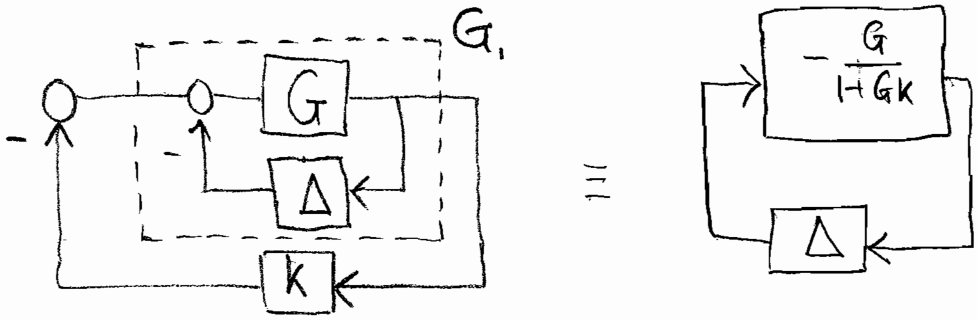
(iv)

	Counter-clockwise encirclements of $-\frac{1}{k}$	# closed-loop poles in RHP
$-7 < k < 18$	0	2
$k > 18$	2	0
$k < -7$	1	1

2(a) Small gain theorem.



Let $F(s)$, $\Delta(s)$ be stable. Feedback loop is stable for all $\Delta(s)$ satisfying $|\Delta(j\omega)| \leq f(\omega)$ if and only if $|F(j\omega)| < 1/f(\omega)$ for all ω.



Hence: k stabilizes all G_1 for which $|\Delta(j\omega)| \leq h(\omega)$ if and only if

$$\left| \frac{G(j\omega)}{1 + G(j\omega)\,k(j\omega)} \right| < \frac{1}{h(\omega)}$$

for all ω.

(b) (i) Write

$$G_1 = \frac{\frac{1}{s^2}}{1 + \frac{a e^{-sT}}{s+3}\,\frac{1}{s^2}} = \frac{G}{1 + \Delta G}$$

where $G = \frac{1}{s^2}$ and $\Delta = \frac{a e^{-sT}}{s+3}$.

$$|\Delta(j\omega)| \leqslant \frac{a_0}{|j\omega + 3|} = f(\omega)$$

$$\frac{G}{1+kG} = \frac{\frac{1}{s^2}}{1 + \frac{3s+1}{s+3} \cdot \frac{1}{s^2}} = \frac{s+3}{s^3+3s^2+3s+1}$$

$$= \frac{s+3}{(s+1)^3}$$

Closed-loop is guaranteed to be stable if

$$\left| \frac{j\omega+3}{(j\omega+1)^3} \right| < \frac{|j\omega+3|}{a_0}$$

for all ω. Require

$$a_0 < |j\omega+1|^3 \quad (\text{all } \omega)$$

i.e. $\quad a_0 < 1$

(no condition on T.)

(b)(ii) The actual class of uncertain $\Delta(s)$ is of the form $\frac{a e^{-sT}}{s+3}$, i.e. it is not a general $\Delta(s)$ satisfying $|\Delta(j\omega)| \leqslant f(\omega)$. Thus the result of (b)(i) may only be sufficient and not necessary.

Following the hint let $T=0$.

$$G_1 = \frac{s+3}{s^2(s+3)+a}$$

Closed-loop poles with $K(s)$ are roots of:

$$\left(s^2(s+3)+a\right)(s+3) + (s+3)(3s+1)$$

$$= (s+3)\left[s^3+3s^2+3s+1+a\right]$$

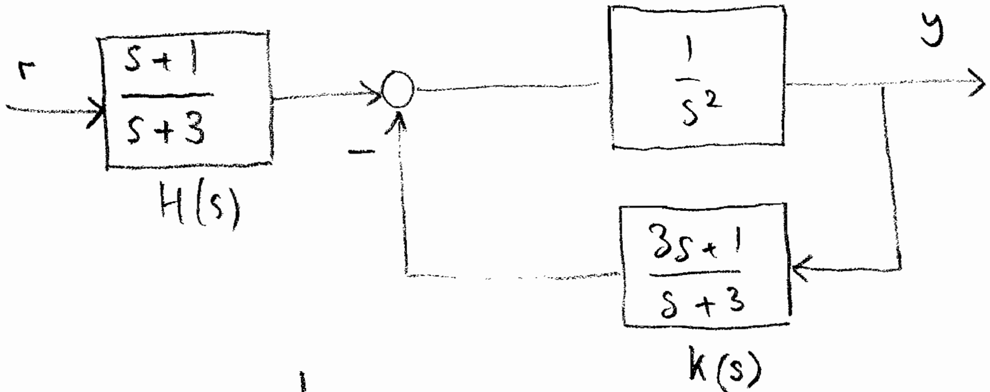
Routh-Hurwitz test: stable $\Longleftrightarrow$ $0 < 1+a < 9$

$\Longleftrightarrow$ $-1 < a < 8$

This is a bigger range for a than found in (b)(i). But it follows that: $G_1(s)$ as in (1) with $K(s)$ is closed-loop stable for all $T \geq 0$ and $|a| \leq a_0$ if and only if $a_0 < 1$.

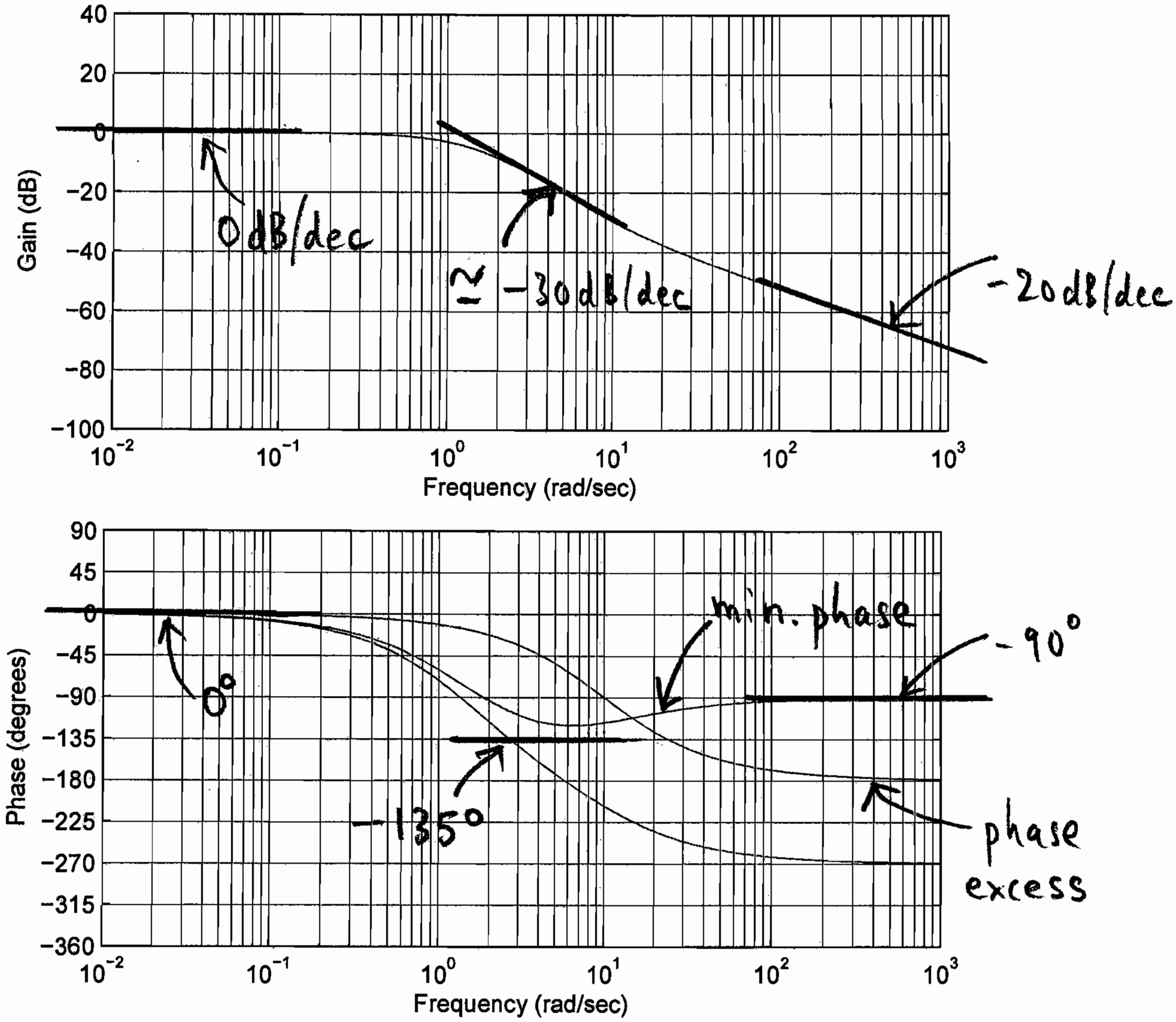
Thus, the condition of part (b)(i) is actually necessary and sufficient when expressed in terms of a_0.

(c)



$$T_{\bar{r}\to\bar{y}} = \frac{1}{(s+1)^2}$$

3 (a) (i)



Straight line approximations to magnitude allow approximate min. phase to be estimated.

(ii) Phase excess decreases from 0° to −180° suggesting the presence of a single RHP zero. Phase goes through −90° around $\omega = 10$ rad/s suggesting a zero at $s = 10$.

[Actual plant: $\dfrac{-0.1s + 1}{0.4225s^2 + 1.3s + 1}$ not needed.]

(b) $\measuredangle G(j2) \simeq -115$ and $|G(j2)| \simeq 0.4$
so $K(s) = 2.5$ satisfies A and B with PM $\simeq 65^\circ$.
To satisfy C need integral action. Choose

$$K_1(s) = 2.5\left(\frac{s+0.2}{s}\right)$$

The zero is placed a decade below the desired crossover frequency ($w=2$) so not to reduce the PM too much (though there is 20° in reserve).

To satisfy D need:

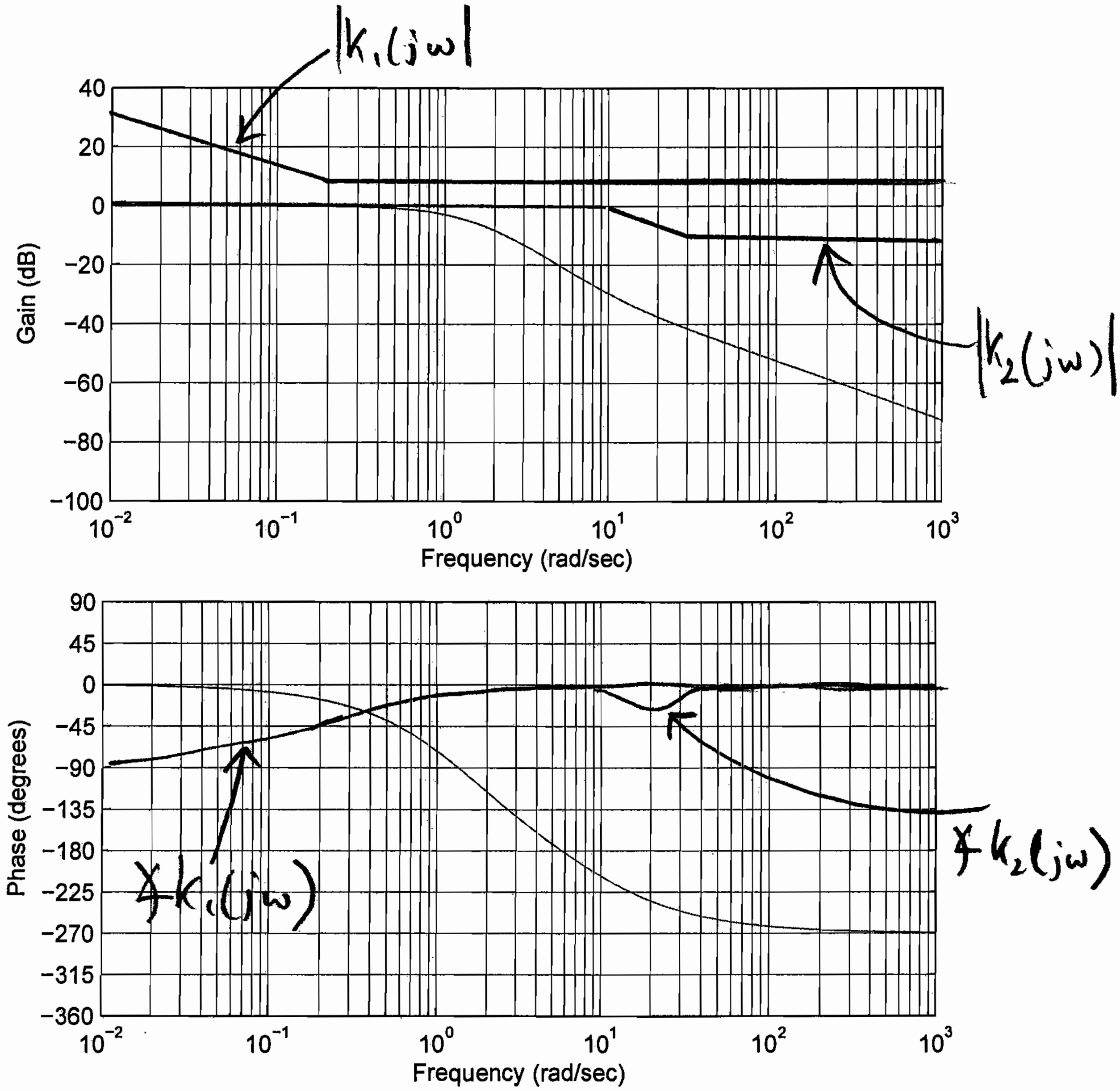
$$|G(jw)K(jw)| \leq 0.01 \qquad (*)$$

(approx.) for $w \geq 30$ rad/sec.

The effect of the extra gain of 2.5 means (*) is not satisfied with $K = K_1$. We need to cut the loop gain at $w = 30$ rad/sec while leaving it unchanged at $w=2$ rad/sec. A lag compensator is needed for this. We need to be careful not to produce too much phase lag at $w=2$ so we still meet spec. B:

$$K_2(s) = \frac{s+30}{3(s+10)}$$

has close to unity gain at $w=2$ and small phase lag, and a gain of $\simeq \frac{1}{3}$ at $w=30$ rad/sec.

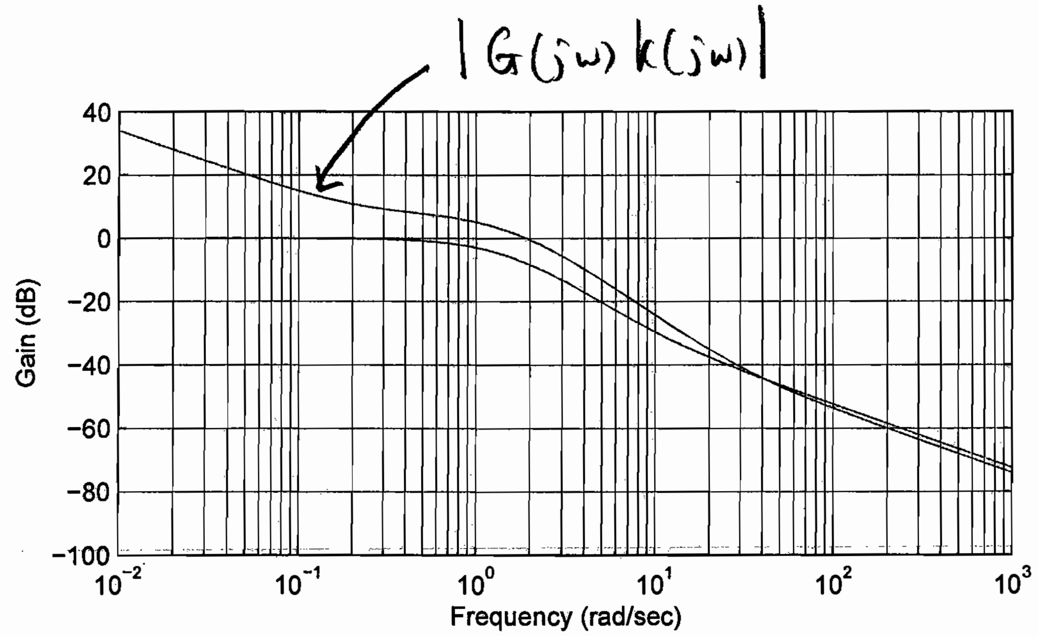


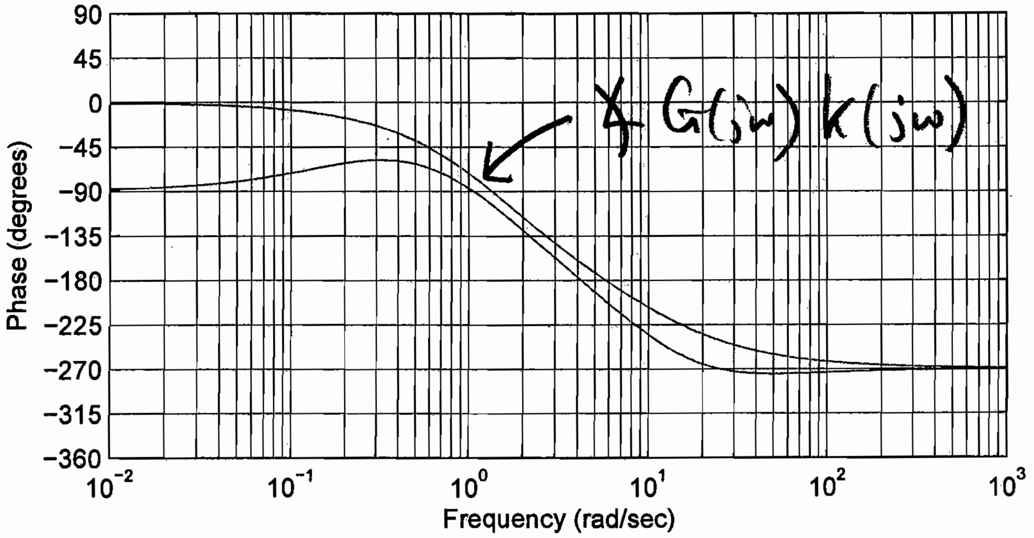
Final compensator

$$K(s) = K_1(s)\, K_2(s)$$

meets all specs (approximately).

Computer generated plot of final $G(s)K(s)$:

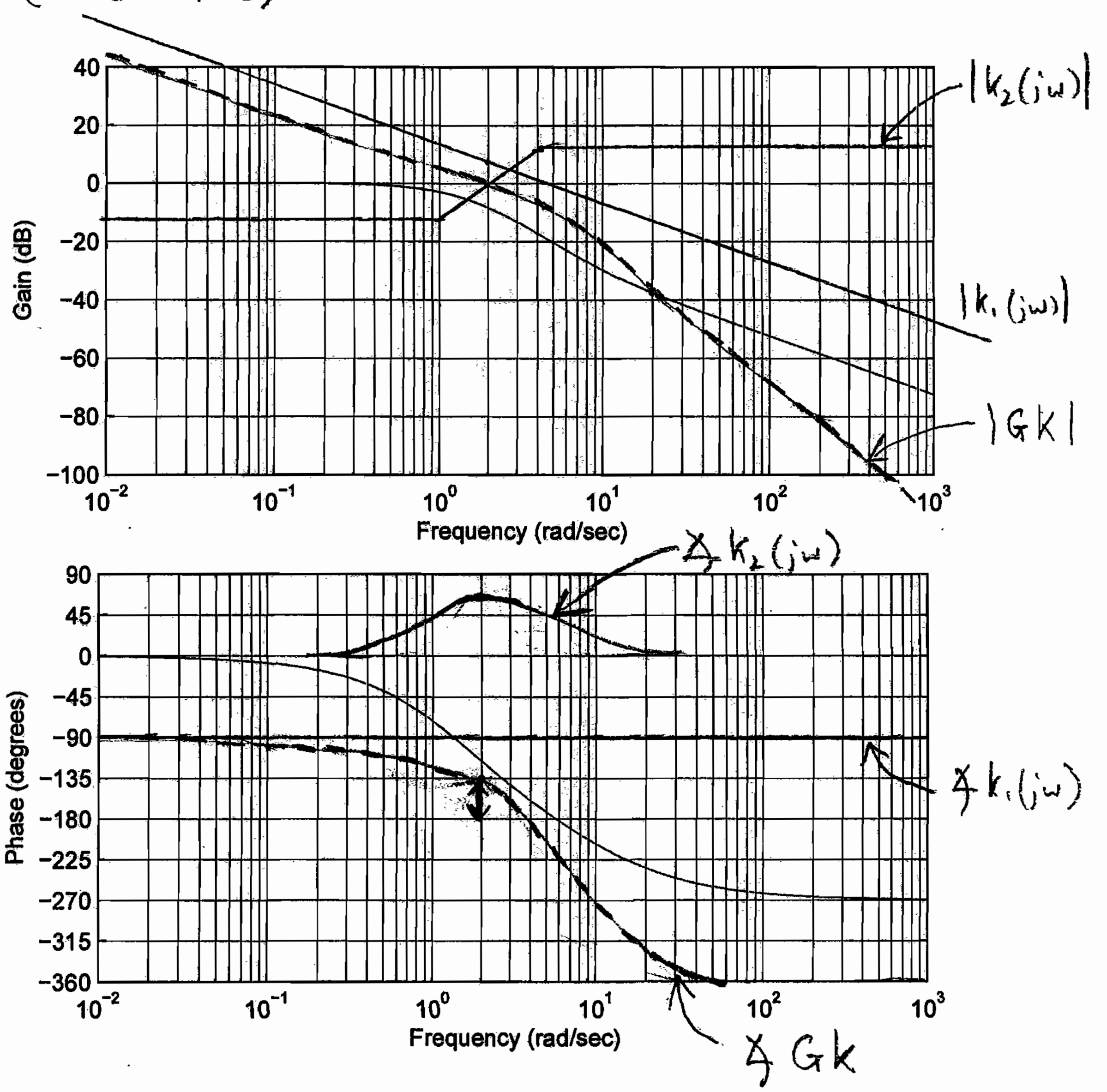




(c)(i) To increase the crossover frequency from $w = 2$ would require a bigger reduction in gain before $w = 30$ to satisfy D which would produce extra phase lag at the new crossover. Would be difficult to increase much! $w = 3$ might be possible but not much more.

(ii) Without spec. D the limitation comes from the RHP zero at $w \simeq 10$. It would be difficult to achieve a crossover much beyond this.

3(b) (Alternative)



Consider first a pure integrator: $k(s) = k_1(s) = \frac{5}{s}$ deals with A, C and D. But we now need to add some phase lead (about 70°) to satisfy B. Try a double lead with $\alpha = 2$ ($\Rightarrow 36.9° \times 2$) centred at $\omega = 2$ rad/sec: $k_2(s) = \left(2 \ \frac{s+1}{s+4}\right)^2$. This increases the high frequency gain so we need to check D again. Let $k = k_1 k_2$. $|k(j30)| \simeq \frac{20}{30} < 1$ so we are o.k. and all specs are met.

Principal Assessor's Report on 4F1 Module Exam
Control System Design

Question 1. Part (a) consisted of bookwork on Nyquist and root-locus diagrams and conformal mappings and was generally answered adequately. Part (b) required sketches of Nyquist and root-locus diagrams for the given plant and an assessment of closed-loop stability. Quite a few candidates went wrong with some of the diagrams through careless slips and failed to notice a lack of consistency between the diagrams. Few candidates successfully found the imaginary and real axis crossing points.

Question 2. Part (a) was a standard piece of bookwork on the small-gain theorem and robustness which was answered well by most candidates. Part (b)(i) was tackled well by many candidates but many also failed in part (b)(ii), missing the easy application of the Routh-Hurwitz criterion. Part (c) was mostly well done.

Question 3. This was a standard Bode gain-phase relationship and compensator design question. Many candidates obtained excellent solutions which demonstrated a very good understanding of the material. It was extremely pleasing for the examiner to see a number of different but equally sound and correct strategies for the compensator design.

M.C. Smith, 11 May 2009