Module 4F2, May 2009 – Robust Multivariable Control - Solutions

ENGINEERING TRIPOS PART IIB

Wednesday 29 April 2009 2.30 to 4

Module 4F2

ROBUST MULTIVARIABLE CONTROL

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Single-sided script paper	Engineering Data Book
	CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

4F2 - 2009

Guy-Bart Stan

1 (a) (i) Definition: $\|\hat{G}(s)\|_2 = \sqrt{\int_{-\infty}^{+\infty} \text{Trace}\left((\hat{G}(j\omega))^* \hat{G}(j\omega)\right) d\omega}$ (2)

Interpretation in terms of $\|w\|_2$ and $\|z\|_\infty$: $\|z\|_\infty = \sup_t \sqrt{z^T(t)\, z(t)} \leq \frac{1}{\sqrt{2\pi}} \|\hat{G}(s)\|_2 \|w\|_2$

with $\|w\|_2 = \sqrt{\int_{-\infty}^{+\infty} w^T(t)\, w(t)\, dt}$

(ii) $\|\hat{G}(s)\|_2 = \sqrt{2\pi}\sqrt{\text{trace}(B^T L B)}$ where L is the solution to the Lyapunov equation

$$\boxed{A^T L + L A + C^T C = 0}$$

The Lyapunov equation admits a solution $L \geq 0$ if the system is stable, i.e. if A has all its eigenvalues in the left hand plane. The solution is positive definite, i.e $L > 0$ if in addition the system is observable.

(b) (i) The closed loop system is

$$\begin{cases} \dot{x} = (A - B_2 K)x + B_1 w \\ z = (C_1 - D_{12} K)x \end{cases}$$

$\mathcal{F}_\ell(P, K)$ is the transfer function from w to z for this closed loop system. Therefore (using (a) (ii))

$$\boxed{\|\mathcal{F}_\ell(P,K)\|_2^2 = 2\pi \ \text{trace}(B_1^T L B_1)}$$

where L solves the Lyapunov equation

$$(A - B_2 K)^T L + L(A - B_2 K) + (C_1 - D_{12}K)^T (C_1 - D_{12} K) = 0$$

$$\Leftrightarrow (A - B_2 K)^T L + L(A - B_2 K) + C_1^T C_1 - \cancel{C_1^T D_{12} K} - \cancel{K^T D_{12}^T C_1} + K^T D_{12}^T D_{12} K = 0$$

We know that $D_{12}^T C_1 = 0$ and $D_{12}^T D_{12} = I$
$(\Leftrightarrow C_1^T D_{12} = 0)$

Therefore, we have that L solves the Lyapunov equation

$$\boxed{(A - B_2 K)^T L + L(A - B_2 K) + C_1^T C_1 + K^T K = 0}$$

(d)

(ii) we have to show that

$$\boxed{(A-B_2K)^T(L-X)+(L-X)(A-B_2K)+(K-B_2^TX)^T(K-B_2^TX)=0} \quad (*)$$

If we expand this equation, we obtain

$$\underbrace{(A-B_2K)^TL+L(A-B_2K)+K^TK}_{=-C_1^TC_1 \text{ (by (b) (i))}}$$

$$\underbrace{-A^TX-XA+X^TB_2B_2^TX}_{=C_1^TC_1}+\cancel{K^TB_2^T}X+X\cancel{B_2K}-\cancel{XB_2}K-\cancel{KB_2^T}X$$

$$=-C_1^TC_1+C_1^TC_1=0 \quad \text{O.K.}$$

By (a) (ii), the Lyapunov equation (*) has a positive semi-definite solution $(L-X)\geq 0$ <u>if and only if</u> $(A-B_2K)$ is stable

$(A-B_2K)$ is stable since, by assumption, K is a stabilizing controller

(iii) $\min_{K \text{ stabilizing}} \|\mathcal{F}_\ell(P,K)\|_2=\min_{K \text{ stabilizing}} \sqrt{2\pi}\sqrt{\operatorname{trace}(B_1^TLB_1)}$

By (b) (ii) $\begin{cases}\text{if } L=L^T\geq 0 \text{ is the solution to } (A-B_2K)^TL+L(A-B_2K)+C_1^TC_1+K^TK=0\\ \text{if } X=X^T>0 \text{ is the solution to } XA+A^TX-XB_2B_2^TX+C_1^TC_1=0\end{cases}$

then $L-X\geq 0$

Now, $L-X\geq 0 \Rightarrow B_1^T(L-X)B_1\geq 0$

$\Rightarrow$ (using the hint) $\operatorname{trace}(B_1^T(L-X)B_1)\geq 0$

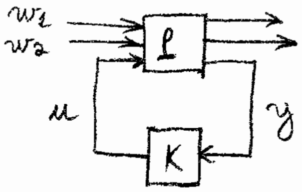
$\Rightarrow \boxed{\operatorname{trace}(B_1^TLB_1)\geq \operatorname{trace}(B_1^TXB_1)}$

Therefore, <u>the minimum is achieved when $L=X$</u>

$\Rightarrow \min_{K \text{ stabilizing}} \|\mathcal{F}_\ell(P,K)\|_2=\sqrt{2\pi}\sqrt{\operatorname{trace}(B_1^TXB_1)}$

and the controller that achieves the minimum is $K=B_2^TX$ from (*) with $L=X$

2 (a)



$$\begin{bmatrix} z_1 \\ z_2 \\ y \end{bmatrix} = \begin{bmatrix} \quad P(s) \quad \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ u \end{bmatrix}$$

From inspection of the block diagram, we have

$$\begin{cases} y = w_2 + z_1 \\ z_1 = \tilde{G} u + w_1 \\ z_2 = u = K y \end{cases} \Rightarrow P(s) = \begin{bmatrix} I & 0 & \tilde{G} \\ 0 & 0 & I \\ I & I & \tilde{G} \end{bmatrix}$$

$\mathcal{F}_\ell(P,K) = T_{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \to \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}}$

$$\Rightarrow y = w_2 + \tilde{G} K y + w_1$$

$$\Leftrightarrow (I - \tilde{G} K) y = w_1 + w_2$$

$$\Rightarrow y = (I - \tilde{G} K)^{-1} (w_1 + w_2)$$

$$\Rightarrow \boxed{z_2 = K (I - \tilde{G} K)^{-1} (w_1 + w_2)}$$

$$z_1 = \tilde{G} K (I - \tilde{G} K)^{-1} (w_1 + w_2) + w_1$$

$$\Leftrightarrow \boxed{z_1 = (I + \tilde{G} K (I - \tilde{G} K)^{-1}) w_1 + \tilde{G} K (I - \tilde{G} K)^{-1} w_2}$$

$$\mathcal{F}_\ell(P, K) = \begin{bmatrix} I + \tilde{G} K (I - \tilde{G} K)^{-1} & \tilde{G} K (I - \tilde{G} K)^{-1} \\ K (I - \tilde{G} K)^{-1} & K (I - \tilde{G} K)^{-1} \end{bmatrix}$$

$\tilde{G}(s) = \dfrac{k}{s - a}$ (unstable pole at $a > 0$)

(b) $G \begin{cases} \dot{x} = a x + k u + w_1, \; x \in \mathbb{R} \\ z_1 = x \quad , a > 0, \; k > 0 \end{cases}$ $\quad z_2 = u \quad$ (w_1 is a process noise) $\quad y = x + w_2$

State-space realisation for $P(s)$

$$\begin{bmatrix} \dot{x} \\ z_1 \\ z_2 \\ y \end{bmatrix} = \left[\begin{array}{c|cc|c} a & \overset{B_1}{1} & 0 & \overset{B_2}{k} \\ \hline \underset{C_1}{+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \underset{C_2}{1} & 0 & 1 & 0 \end{array} \right] \begin{bmatrix} x \\ w_1 \\ w_2 \\ u \end{bmatrix}$$

The problem is of $\mathcal{H}_2$ optim. control (output feedback) with $\begin{cases} B_1 = 1, \; B_2 = k, \; C_1 = 1, \; C_2 = 1 \\ A = a \end{cases}$

④

(c) $\min_{K(s)\text{ stabilising}} \|\mathcal{F}_\ell(P(s), K(s))\|_2 = \sqrt{2\pi}\sqrt{\operatorname{trace}(B_1^T X B_1) + \operatorname{trace}(F Y F^T)}$

where $F = B_2^T X$

X is the stabilising solution of CARE:

$$① \quad XA + A^T X + C_1^T C_1 - X B_2 B_2^T X = 0$$

Y is the stabilising solution of FARE:

$$② \quad YA^T + AY + B_1 B_1^T - Y C_2^T C_2 Y = 0$$

CARE ① $\quad 2ax^* + 1 - k^2 x^{*2} = 0 \Leftrightarrow k^2 x^{*2} - 2ax^* - 1 = 0$

$$\Rightarrow \boxed{x^* = \frac{a + \sqrt{a^2 + k^2}}{k^2}}$$

(the stabilising solution is the >0 one $\forall a, k > 0$)

FARE ② $\quad + y^{*2} - 2a y^* - 1 = 0 \Rightarrow \boxed{y^* = a + \sqrt{a^2 + 1}}$

$$F = B_2^T X = k x^*$$

$$\min_{K(s)\text{ stab}} \|\mathcal{F}_\ell(P(s), K(s))\|_2 = \sqrt{2\pi}\sqrt{\operatorname{trace}(B_1^T X B_1) + \operatorname{trace}(F Y F^T)}$$

$$\boxed{\begin{aligned} B_1^T X B_1 &= x^* \\ F Y F^T &= k^2 x^{*2} y^* \end{aligned}}$$

$$\min_{K(s)\text{ stab}} \|\mathcal{F}_\ell(P(s), K(s))\|_2 = \sqrt{2\pi}\sqrt{x^* + k^2 x^{*2} y^*}$$

$$= \sqrt{2\pi}\sqrt{x^*(1 + k^2 x^* y^*)}$$

$$= \sqrt{2\pi}\sqrt{\frac{(a + \sqrt{a^2+k^2})}{k^2}\left(1 + \cancel{k^2}\frac{(a+\sqrt{a^2+k^2})(a+\sqrt{a^2+1})}{\cancel{k^2}}\right)}$$

$$= \frac{\sqrt{2\pi}}{k}\sqrt{a + \sqrt{a^2+k^2}}\sqrt{\left(1 + (a+\sqrt{a^2+k^2})(a+\sqrt{a^2+1})\right)}$$

(d) The optimal controller K has the following state space realization

$$\begin{bmatrix} \dot{x}_K \\ u \end{bmatrix} = \left[\begin{array}{c|c} A - B_2 F - H C_2 & -H \\ \hline F & 0 \end{array}\right] \begin{bmatrix} x_K \\ y \end{bmatrix}$$

with $F = B_2^T X = k x^*$, $\quad x^* = \dfrac{a + \sqrt{a^2 + k^2}}{k^2}$

$H = Y C_2^T = y^*$, $\quad y^* = a + \sqrt{a^2 + 1}$

$B_2 = k$

$C_2 = 1$

$$\Rightarrow \begin{bmatrix} \dot{x}_K \\ u \end{bmatrix} = \left[\begin{array}{c|c} a - k^2 x^* - y^* & -y^* \\ \hline k x^* & 0 \end{array}\right] \begin{bmatrix} x_K \\ y \end{bmatrix}$$

(e) The poles of the closed loop system are the eigenvalues of $\underbrace{A - B_2 F}$ and those of $\underbrace{A - HC_2}$

$$\begin{aligned} &= a - k^2 x^* \\ &= a - \frac{\cancel{k^2}}{\cancel{k^2}}\left(a + \sqrt{a^2 + k^2}\right) \\ &= -\sqrt{a^2 + k^2} < 0 \quad \forall a, k \\ &\quad \Downarrow \\ &\quad \text{stable} \end{aligned}$$

$$\begin{aligned} &= a - y^* \\ &= \cancel{a} - \cancel{a} - \sqrt{a^2 + 1} \\ &= -\sqrt{a^2 + 1} < 0 \quad \forall a, k \\ &\quad \Downarrow \\ &\quad \text{stable} \end{aligned}$$

<u>Closed-loop system</u>

$$\left.\begin{aligned} \dot{x} &= a x + k u + w_1 \\ \dot{x}_K &= -y^* y + (a - k^2 x^* - y^*) x_K \\ y &= x + w_2 \\ u &= k x^* x_K \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \dot{x} &= a x + k^2 x^* x_K + w_1 \\ \dot{x}_K &= -y^* x + (a - k^2 x^* - y^*) x_K - y^* w_2 \end{aligned}\right.$$

$$\rightarrow A_{\text{closed-loop}} = \begin{pmatrix} a & k^2 x^* \\ -y^* & a - k^2 x^* - y^* \end{pmatrix}$$

<u>eigenvalues of A</u>: λ_1 & λ_2

$$\begin{aligned} \text{Trace}(A) &= \lambda_1 + \lambda_2 = 2a - k^2 x^* - y^* \\ \text{Det}(A) &= \lambda_1 \lambda_2 = a(a - k^2 x^* - y^*) + k^2 x^* y^* \\ &= (a - k^2 x^*)(a - y^*) \end{aligned}$$

$$\Rightarrow \lambda_1 = a - k^2 x^* (<0) \ \& \ \lambda_2 = a - y^* (<0)$$

3 (a) $P(s)$ is in $\mathscr{H}_\infty$ since all poles lie in the open left half plane.

$$\begin{aligned}\|P(s)\|_\infty &= \sup_\omega |P(j\omega)| \\ &= \sup_\omega \left|\frac{j\omega+1}{(j\omega+2)^2}\right| \\ &= \sup_\omega \frac{\sqrt{1+\omega^2}}{4+\omega^2}\end{aligned}$$

Differentiating with respect to w^2 and equating to zero gives

$$\frac{(4+\omega_0^2)\frac{1}{2}(1+\omega_0^2)^{-1/2}-\sqrt{1+\omega_0^2}}{(4+\omega_0^2)^2}=0$$

or $4+\omega_0^2=2(1+\omega_0^2)$, which gives $\omega_0^2=2$. Thus,

$$\|P(s)\|_\infty=\frac{\sqrt{3}}{6}$$

[10%]

(b) The small gain theorem for the robust stability of a system states that the feedback uncertain system is stable for all $\|\Delta(s)\|_\infty<\varepsilon$ if and only if $\|T(s)\|_\infty\le 1/\varepsilon$. [10%]

(c) Let $X=G_0U$, Then $Y=(I-W\Delta)^{-1}X$, or $Y=W\Delta Y+X$. Let also $Z=\Delta Y$. We need to find the transfer function from Z to Y. From above, $Y=WZ-G_0Y$ or $Y=(I+G_0)^{-1}WZ$. Thus, let $T=(I+G_0)^{-1}W$ and the result follows from the small gain theorem as stated in part (b). [30%]

(d) (i) The feedback system is stable if and only if $(1+G(s))^{-1}$ is stable. In this case

$$(1+G(s))^{-1}=\frac{(s+1)(s+2+\alpha)}{s^2+(4+\alpha)s+4+\alpha}$$

which is stable if and only if $\alpha>-4$, so the maximum value of A such that the feedback system is guaranteed to be stable for all $|\alpha|<A$ is $A=4$. [10%]

(ii) We have that $G(s)=(1-W\Delta)^{-1}G_0$. Rearranging gives $(1-W\Delta)G=G_0$ or

$$W\Delta=\frac{G-G_0}{G}=1-\frac{G_0}{G}=1-\frac{s+2+\alpha}{s+2}=-\frac{\alpha}{s+2}$$

(cont.

Let $W = A/(s+2)$ and $\Delta = -\alpha/A$ (which results in $\|\Delta\|_\infty < 1$). Stability condition is

$$\left\| \frac{W}{1+G_0} \right\|_\infty \leq 1$$

or

$$\left\| \frac{A}{s+2} \cdot \frac{s+1}{s+2} \right\|_\infty = A\frac{\sqrt{3}}{6} \leq 1$$

by part (a). Thus, $A \leq 2\sqrt{3} = 3.4641$. [20%]

(iii) We have $G_0 = 1/(s+1) = N/M$ with $|N|^2 + |M|^2 = 1$. Let

$$M = \frac{s+1}{s+\sqrt{2}} \quad \text{and} \quad N = \frac{1}{s+\sqrt{2}}$$

Now, $G = G_0/(1-W\Delta)$ and $W\Delta = -\alpha/(s+2)$. So,

$$G = \frac{N}{M\left(1+\frac{\alpha}{s+2}\right)} = \frac{N}{M + \frac{\alpha(s+1)}{(s+2)(s+\sqrt{2})}}$$

Thus, just let

$$\Delta_N = 0, \quad \Delta_M = \frac{\alpha(s+1)}{(s+2)(s+\sqrt{2})}$$

Perturbations measured in the gap metric are equivalent to perturbations to the numerator and denominator in a normalised coprime factorisation:

$$G = (N+\Delta_N)(M+\Delta_M)^{-1}$$

with $\delta_g(G, G_0) = \min \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_\infty$ over all Δ_N, Δ_M in $\mathscr{H}_\infty$ such that $G = (N+\Delta_N)(M+\Delta_M)^{-1}$. Hence, in the above $\delta_g(G, G_0) \leq \|\Delta_M\|_\infty = \left\| \frac{\alpha(s+1)}{(s+2)(s+\sqrt{2})} \right\|_\infty$. [20%]

END OF PAPER