4F5 Advanced Wireless Communications, 2009 Crib

Question 1

(a) *Consider a pair of discrete random variables X, Y where $X \in \{a,b,c\}$ and $Y \in \{a,b,c\}$ Obtain $H(X), H(Y), H(X|Y), H(Y|X), H(X,Y)$ and $I(X;Y)$ when their joint probabilities are:*

$$P_{X,Y}(X=a,Y=a) = P_{X,Y}(X=b,Y=b) = P_{X,Y}(X=c,Y=c) = \frac{1}{6}$$
$$P_{X,Y}(X=a,Y=b) = P_{X,Y}(X=a,Y=c) = \frac{1}{12}$$
$$P_{X,Y}(X=b,Y=a) = P_{X,Y}(X=b,Y=c) = \frac{1}{12}$$
$$P_{X,Y}(X=c,Y=a) = P_{X,Y}(X=c,Y=b) = \frac{1}{12}$$

In order to calculate the required quantities, we need to calculate the marginal probabilities.

$$\begin{aligned} P_X(x) &= \sum_y P_{X,Y}(x,y) \\ &= P_{X,Y}(x,Y=a) + P_{X,Y}(x,Y=b) + P_{X,Y}(x,Y=c) \\ P_X(X=a) &= \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{1}{3} \\ P_X(X=b) &= \frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{1}{3} \\ P_X(X=c) &= \frac{1}{12} + \frac{1}{12} + \frac{1}{6} = \frac{1}{3} \end{aligned}$$

Similarly, we obtain

$$
\begin{aligned}
P_Y(y) &= \sum_x P_{X,Y}(x,y) \\
&= P_{X,Y}(X=a,y) + P_{X,Y}(X=b,y) + P_{X,Y}(X=c,y) \\
P_Y(Y=a) &= \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{1}{3} \\
P_Y(Y=b) &= \frac{1}{12} + \frac{1}{6} + \frac{1}{12} = \frac{1}{3} \\
P_Y(Y=c) &= \frac{1}{12} + \frac{1}{12} + \frac{1}{6} = \frac{1}{3}
\end{aligned}
$$

Using the above results we obtain

$$
\begin{aligned}
H(X) &= -\sum_x P_X(x) \log_2 P_X(x) = 1.59 \text{ bits} \\
H(Y) &= -\sum_y P_Y(y) \log_2 P_Y(y) = 1.59 \text{ bits} \\
H(X,Y) &= -\sum_x \sum_y P_{X,Y}(x,y) \log_2 P_{X,Y}(x,y) = 3.08 \text{ bits}
\end{aligned}
$$

From the above quantities, it is easy to obtain all the rest

$$
\begin{aligned}
H(X|Y) &= H(X,Y) - H(Y) = 1.49 \text{ bits} \\
H(Y|X) &= H(X,Y) - H(X) = 1.49 \text{ bits} \\
I(X;Y) &= H(X) - H(X|Y) = 0.10 \text{ bits}
\end{aligned}
$$

(b) *Let Y_1 be the output of channel 1 to input X_1 and Y_2 be the output of channel 2 to input X_2 (see Fig. ??). Obtain the mutual information between the input of channel 1 and the output of channel 2, $I(X_1;Y_2)$, when the output of channel 1 to equally likely input symbols is used as input for channel 2.*

The concatenation of the 2 channels is a channel whose alphabet is given by $\mathcal{X} = \{0,1\}$ and $\mathcal{Y} = \{0,1,?\}$. The corresponding channel transition probabilities are

$$
\begin{aligned}
P_{Y|X}(0|0) &= \frac{2}{3} \\
P_{Y|X}(1|1) &= \frac{2}{3} \\
P_{Y|X}(1|0) &= \frac{2}{9} \\
P_{Y|X}(0|1) &= 0 \\
P_{Y|X}(?|0) &= \frac{1}{9} \\
P_{Y|X}(?|1) &= \frac{1}{3}
\end{aligned}
$$

In order to calculate the mutual information we need the marginal probabilities $P_Y(y)$. These are calculated below

$$
\begin{aligned}
P_Y(0) &= P_{Y|X}(0|0)P_X(0) + P_{Y|X}(0|1)P_X(1) = \frac{2}{3}\frac{1}{2} = \frac{1}{3} \\
P_Y(1) &= P_{Y|X}(1|0)P_X(0) + P_{Y|X}(1|1)P_X(1) = \frac{2}{9}\frac{1}{2} + \frac{2}{3}\frac{1}{2} = \frac{4}{9} \\
P_Y(?) &= P_{Y|X}(?|0)P_X(0) + P_{?|X}(0|1)P_X(1) = \frac{1}{9}\frac{1}{2} + \frac{1}{3}\frac{1}{2} = \frac{2}{9}
\end{aligned}
$$

Then

$$
\begin{aligned}
I(X;Y) &= -\sum_x \sum_y P_{X,Y}(x,y) \log_2 \frac{P_{Y|X}(y|x)}{P_Y(y)} \\
&= \sum_y P_{X,Y}(0,y) \log_2 \frac{P_{Y|X}(y|0)}{P_Y(y)} + \sum_y P_{X,Y}(1,y) \log_2 \frac{P_{Y|X}(y|1)}{P_Y(y)} \\
&= \sum_y P_{Y|X}(y|0)P_X(0) \log_2 \frac{P_{Y|X}(y|0)}{P_Y(y)} + \sum_y P_{Y|X}(y|1)P_X(1) \log_2 \frac{P_{Y|X}(y|1)}{P_Y(y)} \\
&= \frac{2}{3}\frac{1}{2} \log_2 \frac{\frac{2}{3}}{\frac{1}{3}} + \frac{1}{9}\frac{1}{2} \log_2 \frac{\frac{1}{9}}{\frac{2}{9}} + \frac{2}{9}\frac{1}{2} \log_2 \frac{\frac{2}{9}}{\frac{4}{9}} + 0 + \frac{1}{3}\frac{1}{2} \log_2 \frac{\frac{1}{3}}{\frac{2}{9}} + \frac{2}{3}\frac{2}{2} \log_2 \frac{\frac{1}{3}}{\frac{4}{9}} \\
&= 0.4591 \quad \text{bits}
\end{aligned}
$$

(c) *Fig. ?? shows the random coding error exponents of 2 different coding schemes in solid and dashed lines, respectively). What can be said about their respective error probabilites and capacities?*

From inspection we observe that the system in solid lines has smaller capacity than that with dashed lines. However, for low rates, the error probability of the solid line scheme decays significantly faster than that of the dashed-line scheme.

(d) *Explain why Gaussian signal constellations are impractical. Show that the mutual information for QPSK modulation over the AWGN satisfies that*

$$I(X;Y) \leq 2.$$

The only way to implement Gaussian codebooks is to store the whole codebook (2^{nR} codewords of length n) in a memory and compare the received signal to each codeword by exhaustive search. This procedure is impractical for lengths of practical interest. Furthermore, since the support of the Gaussian constellation is the whole complex plane, there is no reference that can be taken for implementation. Instead, practical constellations like QAM or PSK can be implemented in practice.

The mutual information for a uniform M-ary signal set can be written as

$$\begin{aligned} I(X;Y) &= \mathbb{E}\left[\log_2 \frac{P_{Y|X}(Y|X)}{P_Y(Y)}\right] \\ &= \mathbb{E}\left[\log_2 \frac{P_{Y|X}(Y|X)}{\sum_{x' \in \mathcal{X}} P_{Y|X}(Y|x')P(x')}\right] \\ &= \mathbb{E}\left[\log_2 \frac{P_{Y|X}(Y|X)}{\frac{1}{M}\sum_{x' \in \mathcal{X}} P_{Y|X}(Y|x')}\right] \\ &= \log_2 M - \mathbb{E}\left[\log_2 \frac{\sum_{x' \in \mathcal{X}} P_{Y|X}(Y|x')}{P_{Y|X}(Y|X)}\right] \\ &= \log_2 M - \mathbb{E}\left[\log_2 \left(1 + \sum_{x' \neq X} \frac{P_{Y|X}(Y|x')}{P_{Y|X}(Y|X)}\right)\right] \end{aligned}$$

Since the terms inside the sum over $x' \neq X$ are probabilities, these are all positive. Then, from the properties of the logarithm, we have that $\log_2(1 + A) \geq 0$ if $A \geq 0$. Hence

$$I(X;Y) \leq \log_2 M.$$

Question 2

(a) *Consider the signal set shown in Fig. ??. Find an orthonormal basis of the signal space. What is the dimension of the signal set?*

Using the Gram-Schmidt procedure we obtain

$$f_1(t) = \frac{x_1(t)}{\sqrt{E_1'}}, \quad E_1' = \int x_1^2(t)dt = 2$$

$$f_2(t) = \frac{f_2'(t)}{\sqrt{E_2'}}, \quad f_2'(t) = x_2(t) - c_{2,1}f_1(t), \quad c_{2,1} = 0, \quad E_2' = \int x_2^2(t)dt = 2$$

$$f_3(t) = \frac{f_3'(t)}{\sqrt{E_3'}}, \quad f_3'(t) = x_3(t) - c_{3,1}f_1(t) - c_{3,2}f_2(t),$$

$$c_{3,1} = \int x_3(t)f_1(t)dt = \sqrt{2}, c_{3,2} = 0, E_3' = 1$$

$$f_4(t) = 0$$

The signals are shown in Fig. 1

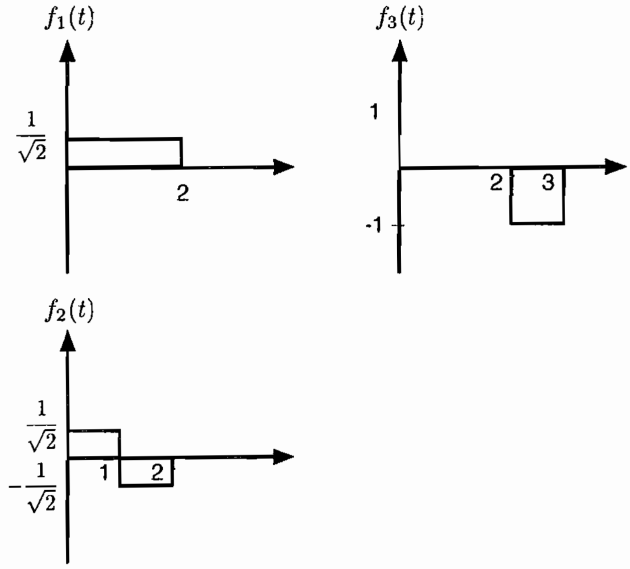

Figure 1: Basis for Question 1.

The dimension of the signal space is $K = 3$.

(b) *Write the signal constellation points and draw the signal constellation.*

The signal constellation points are

$$x_1 = (\sqrt{2}, 0, 0), \; x_2 = (0, \sqrt{2}, 0), \; x_3 = (\sqrt{2}, 0, 1), \; x_4 = (-\sqrt{2}, 0, 1).$$

The 3-D signal constellation is shown in Fig.

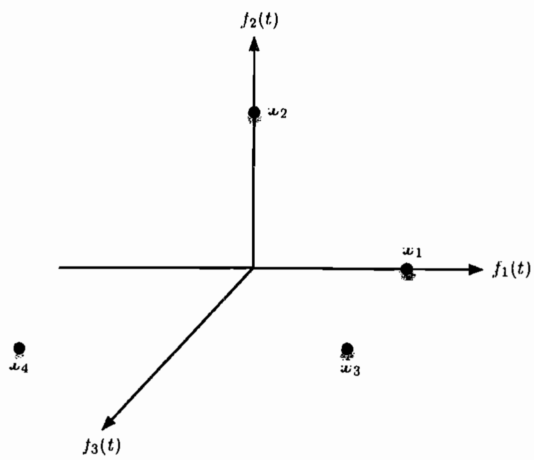

Figure 2: Basis for Question 1.

(c) *What is minimum distance of the constellation?*

The pairwise distances are

$$d_{1,2}^2 = \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^2 = \sum_{k=1}^{3}(x_{1,k} - x_{2,k})^2 = 2 + 2 + 0 = 4$$

$$d_{1,3}^2 = \|\boldsymbol{x}_1 - \boldsymbol{x}_3\|^2 = \sum_{k=1}^{3}(x_{1,k} - x_{3,k})^2 = 0 + 0 + 1 = 1$$

$$d_{1,4}^2 = \|\boldsymbol{x}_1 - \boldsymbol{x}_4\|^2 = \sum_{k=1}^{3}(x_{1,k} - x_{4,k})^2 = 8 + 0 + 1 = 9$$

$$d_{2,3}^2 = \|\boldsymbol{x}_2 - \boldsymbol{x}_3\|^2 = \sum_{k=1}^{3}(x_{2,k} - x_{3,k})^2 = 2 + 2 + 1 = 5$$

$$d_{2,4}^2 = \|\boldsymbol{x}_2 - \boldsymbol{x}_4\|^2 = \sum_{k=1}^{3}(x_{2,k} - x_{4,k})^2 = 2 + 2 + 1 = 5$$

$$d_{3,4}^2 = \|\boldsymbol{x}_3 - \boldsymbol{x}_4\|^2 = \sum_{k=1}^{3}(x_{3,k} - x_{4,k})^2 = 8 + 0 + 0 = 8$$

Hence the minimum distance is $d_{\min} = d_{1,3} = 1$.

(d) *BPSK modulation is employed to transmit over an AWGN channel. The channel introduces a phase rotation of* 45 *degrees, and this phase is known at the* *receiver. How does this affect the signal constellation using the standard basis? Does it affect the error probability?*

The effect translates into a 45 degree rotation of the signal constellation. This is shown in Fig. 3.

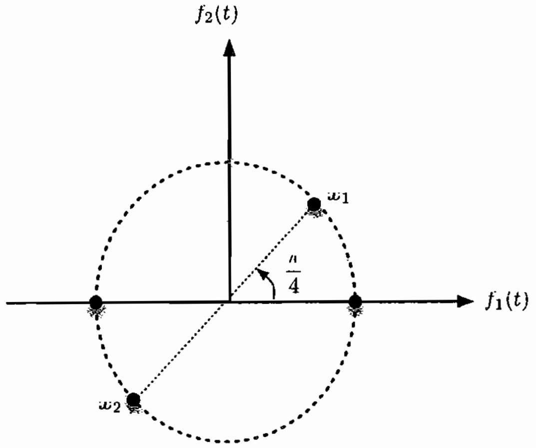

Figure 3: Basis for Question 1.

If the phase shift is known, it does not affect the error probability. It is sufficient to rotate the signal space basis functions by 45 degrees to recover the original BPSK signal set and hence the standard error probability calculation applies.

Question 3

(a) *Consider a* $(8,4)$ *linear binary code, with codewords on the form* $(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4)$ *where* $x_i, i = 1, 2, 3, 4,$ *are the information bits and* $p_i, i = 1, 2, 3, 4,$ *are the parity check bits. The code is defined as*

$$p_1 = x_2 + x_3 + x_4$$
$$p_2 = x_1 + x_2 + x_3$$
$$p_3 = x_1 + x_2 + x_4$$
$$p_4 = x_1 + x_3 + x_4$$

(where + denotes modulo 2 addition). Give a generator matrix, a corresponding parity check matrix. What is the rate of the code?

The generator matrix can be written as

$$\boldsymbol{G} = [\boldsymbol{I} \mid \boldsymbol{P}] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

since the code is described in systematic form. The corresponding parity check matrix is then

$$\boldsymbol{H} = [\boldsymbol{P}^T \mid \boldsymbol{I}] = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The code has rate $R = 1/2$, since it has 4 information bits and it adds 4 parity bits to the codeword.

(b) *Find the minimum distance.*

The minimum distance is obtained as the minimum number of columns of $\boldsymbol{H}$ required to sum to the zero vector. From the result in (a) we see that $d_{\text{min}} = 4$.

(c) *Draw the factor graph describing the above code. Interpreted as a low-density parity check code, write the variable and check-node degree distribution polynomials of the code (edge perspective).*

The graph is shown in Fig. 4.

The code is check-regular with all edges connected to nodes of degree 4. We have 4 nodes with degree 3 and other 4 with degree 1. Hence, the edge-perspective distribution polynomials are

$$\lambda(x) = \frac{1}{4} + \frac{3}{4}x^2 \qquad \rho(x) = x^3.$$

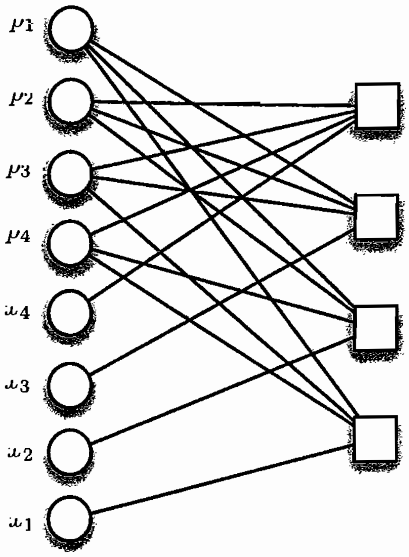

Figure 4: Factor graph o the code in Question 3.

(d) *The above code is used for transmission over a binary erasure channel. Can the iterative decoder decode successfully if x_2, x_4 and p_3 are erased while all other bits are received correctly?*

By removing the edges corresponding to known bits, we are left with the graph shown in Fig. 5. We readily see that from the 4th check node, bit p_3 can be recovered, and hence x_4. x_2 can be similarly recovered as the check node to which it is connected has degree 1.

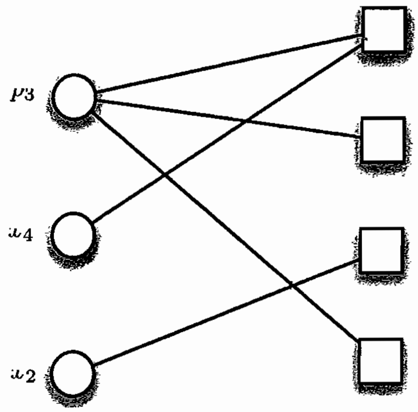

Figure 5: Factor graph o the code in Question 3.

Question 4

(a) *Consider the convolutional code generated by the encoder in Fig. ??. What are the code generators in octal form and what is the rate?*

The code has rate $R = 1/2$ and the generators are $(15, 17)_8$.

(b) *Draw a section of the trellis diagram specifying clearly the contents of the memory in each state and the input and the output corresponding to each transition. Find the free distance, d_{free}, of the code.*

The trellis section is shown in Fig. 6

The free distance can be found by tracking the lowest weight path leaving state S_0 and coming back to it. For this code the path is easily seen to be $S_0 \to S_4 \to S_6 \to S_3 \to S_1 \to S_0$ which gives $d_{\text{free}} = 6$.

(c) *What is the diversity achieved by the code when transmitted using BPSK modulation over a fully-interleaved Rayleigh fading channel? Compare with uncoded BPSK modulation.*

According to pairwise error probability analysis, the diversity is given by the minimum distance of the code, which in this case is the free distance $d_{\text{free}} = 6$.

(d) *What is the diversity achieved by the code when transmitted using bit-interleaved coded modulation with 16-QAM over a fully-interleaved Rayleigh fading channel? Justify your answer.*

The diversity is the same, as BICM preserves the properties of the underlying binary code.

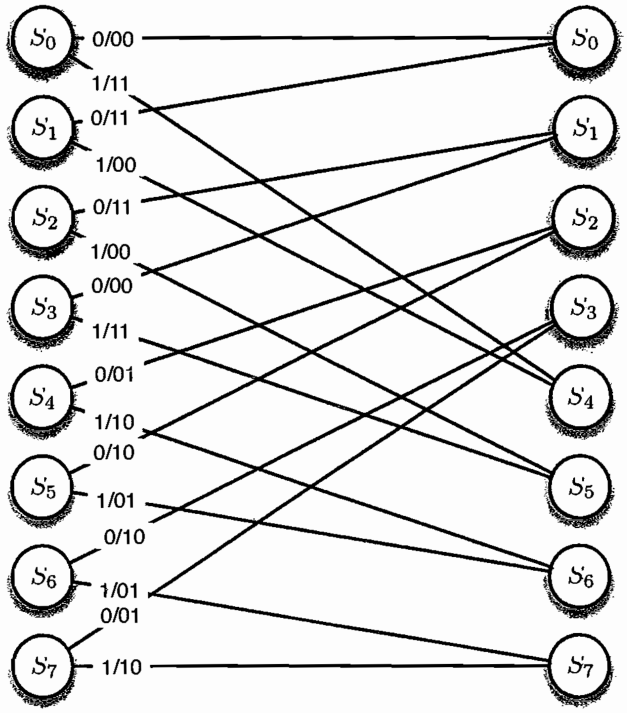

Figure 6: Trellis section for Questions 4.

(e) *Draw the block-diagram of a parallel turbo-code of rate $R = 1/3$ using the recursive encoder corresponding to the feedforward encoder shown in Fig.* ??.
The corresponding block-diagram is shown in Fig. 7.

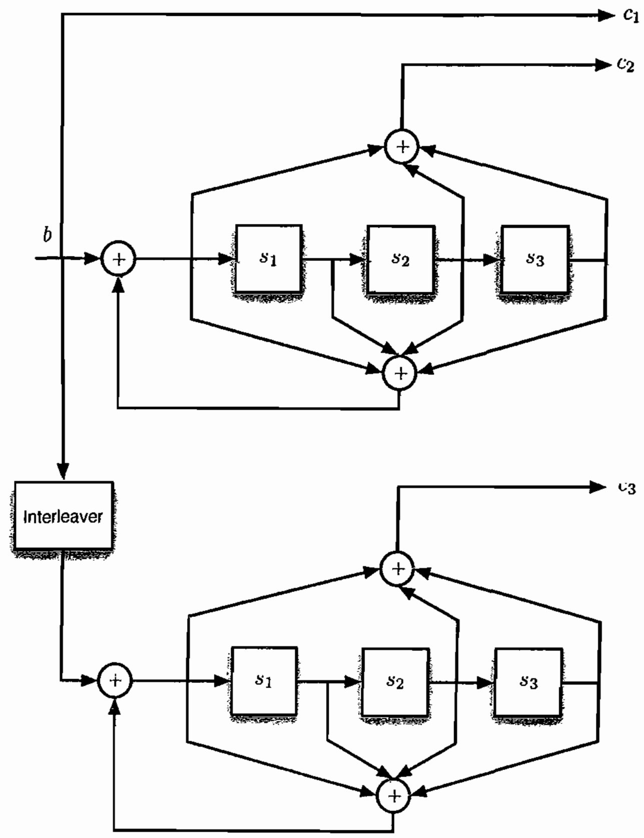

Figure 7: Parallel turbo-code using recursive encoders.