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Q1 a)

$$\varepsilon_{ijk} S_{jk} = \frac{1}{2}\left(\varepsilon_{ijk} S_{jk} + \varepsilon_{ijk} S_{jk}\right)$$
$$= \frac{1}{2}\left(\varepsilon_{ijk} S_{jk} + \varepsilon_{ikj} \cancel{S_{jk}}_{kj}\right) \quad (\text{sym}$$
$$= \frac{1}{2}\left(\varepsilon_{ijk} S_{jk} - \varepsilon_{ijk} S_{jk}\right)$$
$$\therefore \quad S_{ij} = S_{ji} \quad (\text{symmetry})$$

b) $(\underline{a} \times \underline{b}) \times \underline{c}$

$$\cancel{\varepsilon_{ijk} a_j b_k \times [c}]$$
$$\varepsilon_{ijk} (\underline{a} \times \underline{b})_j c_k$$
$$= \varepsilon_{ijk} (\varepsilon_{jlm} a_l b_m) c_k$$
$$= \delta_i$$
$$= \varepsilon_{kij} \varepsilon_{jlm} \{ a_l b_m c_k$$
$$= (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il}) a_l b_m c_k$$
$$= \delta_{kl} a_l b_i c_k - \delta\, b_k \delta_{il} a_l c_k$$
$$b_k c_k a_i$$
$$= a_k c_k b_i - b_k c_k a_i$$
$$\therefore \quad \beta = \underline{a}\cdot\underline{c}, \quad \alpha = \underline{b}\cdot\underline{c}.$$

c) i) $\nabla\cdot(\underline{a}\phi) = \cancel{\nabla\cdot\underline{a}}^{\,0}\phi + \underline{a}\nabla\phi$

ii)
$$\int_V w\underline{a}\cdot\nabla\phi \, dV = -\int_V \nabla\cdot(w\underline{a})\phi\, dV + \int_S w\underline{a}\phi\cdot\underline{n}\, dS$$
$$= -\int \nabla w\cdot\underline{a}\phi\, dV + \int w\phi\underline{a}\,\underline{n}\, dS$$
$$\phi\underline{a}\cdot\underline{n} = 0 \rightarrow \underline{a}\cdot\underline{n} = 0$$

d) $\int_{S} n_1 \, dS = \int \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \underline{n} \, dS$

$= \int_V \nabla \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} dV = 0$

Likewise for the other components

Q2 a) $I = \frac{1}{2}\int_V \nabla u \cdot \nabla u \, dV + \frac{1}{2}\int_V k u^2 \, dV - \int_V f u \, dV$

$$DI(u)[v] = \frac{d}{d\varepsilon}\Big[\frac{1}{2}\int_V (\nabla u + \nabla(\varepsilon v))(\nabla u + \nabla(\varepsilon v))\, dV$$

directn deriv.

$$+ \frac{1}{2}\int_V k^2 (u+\varepsilon v)^2 \, dV$$

$$- \int_V f(u+\varepsilon v)\, dV \Big]\Big|_{\varepsilon=0}$$

$$= \int_V (\nabla u + \nabla(\varepsilon v))\nabla v \, dV + \int_V k^2(u+\varepsilon v) v - \int f(\cancel{\;}\Big|_{\varepsilon=0}$$

$$= \int_V \nabla u \cdot \nabla v + k^2 u v \, dV - \int_V f v \, dV \quad \forall v$$

b) Integrd by parts, $-\nabla^2 u v = \nabla u \nabla v - v \nabla u \cdot n \, dS$

$$-\int_V \nabla^2 u \, v \, dV + \int_S v \nabla u \cdot n \, dS + \int_V k^2 u v - \int f v \, dV$$

Since this must hold for all v

$$\int (\underbrace{-\nabla^2 u + k^2 u - f}_{PDE.}) v = 0$$

c) $I = \frac{1}{2}\int \nabla u \cdot \nabla u \, dV + \frac{1}{2}\int k u^2 \, dV - \int f u \, dV - \int_{S_h} h u \, dS$

Weak for

$$\frac{1}{2}\int \nabla u \cdot \nabla u + k u^2 \, dV - \int_V f u \, dV - \int_{S_h} h u \, dS$$

$$\int \nabla u \cdot \nabla v + k^2 u v - \int_V f v \, dV - \int_{S_h} h v \, dS$$

Integrate by parts

$$-\int \nabla^2 u \, v + k^2 u v \, dV + \int v \nabla u \cdot n \, dS$$
$$- \int_V f v - \int h v \, dS$$

$$\Rightarrow \int (-\nabla^2 u + k^2 u - f) v$$
$$+ \int_{S_h} v \nabla u \cdot n - \int_S v h \, dS$$

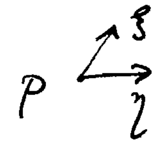
$$\therefore \quad \nabla u \cdot n = h \quad \text{on } S_h$$

3a) For an equation of the form (where $b^2 - ac > 0$)

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

two independent families of curves ($\xi = \text{const}$ & $\eta = \text{const}$) can be found such that, at any point P, the equation becomes

$$\frac{\partial^2 u}{\partial \xi \partial \eta} + \cdots = 0 . \qquad (*)$$



The equations can be integrated along these directions (so called characteristics) as a pair of ode's.

[The condition $b^2 - ac > 0$ is necessary to ensure that these directions are real — such equations are refered to as hyperbolic].

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b) $\xi = y - \frac{x^2}{2} \quad \eta = y \quad \Rightarrow \quad x = \sqrt{2(\eta - \xi)} \quad y = \eta$

$$\frac{\partial x}{\partial \xi} = +\frac{1 \cdot -2}{2[2(\eta-\xi)]^{1/2}} = -\frac{1}{x} \qquad \frac{\partial x}{\partial \eta} = \frac{1}{x}$$

$$\frac{\partial y}{\partial \xi} = 0 \qquad \frac{\partial y}{\partial \eta} = 1$$

$$\Rightarrow \frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \eta}\frac{\partial}{\partial y}$$

i.e. $\dfrac{\partial}{\partial \xi} = -\dfrac{1}{x}\dfrac{\partial}{\partial x} \qquad \dfrac{\partial}{\partial \eta} = \dfrac{1}{x}\dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}$

Now $$\frac{\partial^2 u}{\partial \xi \partial \eta} = -\frac{1}{x}\frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = -\frac{1}{x^2}\frac{\partial^2 u}{\partial x^2} + \frac{1}{x^3}\frac{\partial u}{\partial x} - \frac{1}{x}\frac{\partial^2 u}{\partial x \partial y}$$

$$= -\frac{1}{x^2}\left(-x^2\right) = 1$$

Thus in these variables eq (1) becomes the standard form (*) and hence ξ & η are characteristic variables.

(c) $\dfrac{\partial^2 u}{\partial \xi \partial \eta} = +1 \Rightarrow \dfrac{\partial u}{\partial \eta} = \xi + A(\eta) \Rightarrow u = \xi\eta + F(\xi) + G(\eta)$

The general solution is as

$$u = \left(y - \frac{x^2}{2}\right)y + F\left(y - \frac{x^2}{2}\right) + G(y)$$

for arbitrary functions F & G.

$u(1,y) = 0 \Rightarrow \left(y - \frac{1}{2}\right)y + F\left(y - \frac{1}{2}\right) + G(y) = 0$ (a)

$\dfrac{\partial u}{\partial x}(1,y) = -y-1 \Rightarrow -xy + F'\left(y - \frac{x^2}{2}\right)(-x) = -y-1$ on $x = 1$

i.e. $-y - F'\left(y - \frac{1}{2}\right) = -y - 1$

$\Rightarrow F' = 1 \Rightarrow F(\xi) = \xi$

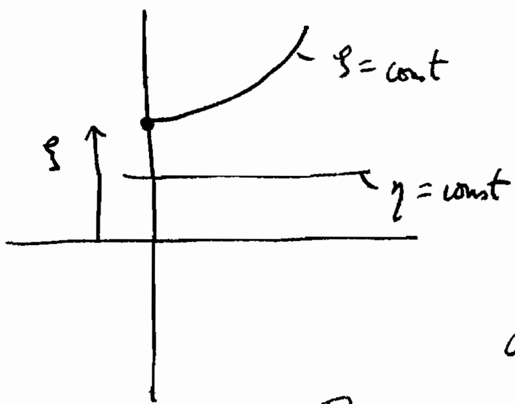
Returning to (a) $\left(y - \frac{1}{2}\right)y + y - \frac{1}{2} + G(y) = 0$

$\Rightarrow G(y) = -y^2 - \frac{y}{2} + \frac{1}{2}$

$$\therefore u(x,y) = \left(y - \frac{x^2}{2}\right)y + y - \frac{x^2}{2} - y^2 - \frac{y}{2} + \frac{1}{2}$$

$$= -\frac{x^2 y}{2} + \frac{y}{2} - \frac{x^2}{2} + \frac{1}{2}$$

(d)



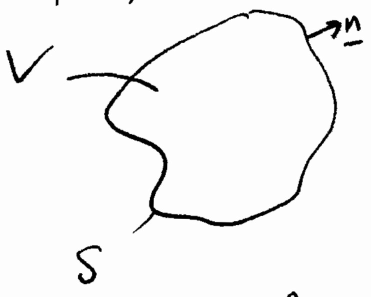
ch'ics are $\xi = $ const $\Rightarrow y = \frac{x^2}{2} + \xi$

& $\eta = $ const $\Rightarrow y = \eta$

These coincide at $x = 0$ (in direction), so problem not well-posed.

or The equation is elliptic only in $x > 0$ & becomes singular & parabolic @ $x = 0$

4 a)



If G is a Green Function of Laplace's eq^n

$$\nabla^2 G = \delta(\underline{x}-\underline{x}_0) \quad \underline{x} \text{ in } V$$

$$\therefore \int (u \nabla^2 G - G \nabla^2 u)\, dV = \int \left(u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n}\right) dS$$

{either by quoting Green's Thm or

$$\int_V (u \nabla^2 G - G\nabla^2 u)\, dV = \int_V \underline{\nabla}.(u\nabla G - G\nabla u)\, dV$$

$$= \int_S (u \nabla G - G\nabla u).\underline{n}\, dS \quad \text{using divergence thm.}\}$$

$$\therefore \int [u(\underline{x})\delta(\underline{x}-\underline{x}_0) - G \times 0]\, dV = \int_S \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) dS$$

& $\int u(\underline{x})\,\delta(\underline{x}-\underline{x}_0)\, dV = u(\underline{x}_0)$ for $\underline{x}_0 \in V$

$$\therefore \quad u(\underline{x}_0) = \int_S \left(u \frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right) dS \qquad \left[\text{See } (A)\right]$$

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b) Using Datacard $\nabla^2 \dfrac{1}{4\pi|\underline{x}-\underline{x}_0|} = \delta(\underline{x}-\underline{x}_0)$

$$\therefore \nabla^2 G = \delta(\underline{x}-\underline{x}_0) - \frac{a}{|\underline{x}_0|}\,\delta(\underline{x}-\underline{x}_1)$$

So for $\underline{x}_0 \in V$ $\underline{x}_1$ is outside $V \Rightarrow \cancel{\delta(\underline{x}-\underline{x}_1)} = 0$.

on $r=a$

$$|\underline{x}-\underline{x}_0|^2 = r^2 + r_0^2 - 2rr_0\cos\alpha = a^2 + r_0^2 - 2ar_0\cos\alpha$$

$$|\underline{x}-\underline{x}_1|^2 = r^2 + \frac{a^4}{r_0^2} - 2r\frac{a^2}{r_0}\cos\alpha = a^2 + \frac{a^4}{r_0^2} - 2\frac{a^3}{r_0}\cos\alpha$$

$$= \frac{a^2}{r_0^2}(a^2 + r_0^2 - 2ar_0\cos\alpha) = \frac{a^2}{r_0^2}|\underline{x}-\underline{x}_0|^2$$

Thus on $r=a$ $\quad |\underline{x}-\underline{x}_1| = \dfrac{a}{|\underline{x}_0|}|\underline{x}-\underline{x}_0|$ $\quad$ (A)

$\Rightarrow \quad G = 0$

$\therefore$ G is the Green fn appropriate to $\nabla^2 u = 0$ in V
with $u = U$ on boundary

(c) $\dfrac{\partial}{\partial n} \equiv \dfrac{\partial}{\partial r}$ on boundary $r=a$

$$\therefore \frac{\partial}{\partial r}|\underline{x}-\underline{x}_0|^2 = 2|\underline{x}-\underline{x}_0|\frac{\partial|\underline{x}-\underline{x}_0|}{\partial r} = 2r - 2r_0\cos\alpha$$

$$\Rightarrow \left.\frac{\partial|\underline{x}-\underline{x}_0|}{\partial r}\right|_{r=a} = \frac{a - r_0\cos\alpha}{|\underline{x}-\underline{x}_0|}$$

$$\frac{\partial}{\partial r}|\underline{x}-\underline{x}_1|^2 = 2|\underline{x}-\underline{x}_1|\frac{\partial|\underline{x}-\underline{x}_1|}{\partial r} = 2r - \frac{2a^2}{r_0}\cos\alpha$$

$$\Rightarrow \left.\frac{\partial|\underline{x}-\underline{x}_1|}{\partial r}\right|_{r=a} = \frac{a - \frac{a^2}{r_0}\cos\alpha}{|\underline{x}-\underline{x}_1|} = \frac{r_0\left(a - \frac{a^2}{r_0}\cos\alpha\right)}{a|\underline{x}-\underline{x}_0|} \text{ using (A)}$$

$$= \frac{r_0 - a\cos\alpha}{|\underline{x}-\underline{x}_0|}.$$

(d) $u(\underline{x}_0) = \displaystyle\int_S U(\underline{x})\frac{\partial G}{\partial n}\,dS$ $\quad$ & $\quad \dfrac{\partial G}{\partial n} = \dfrac{\partial G}{\partial r}$

$$\therefore \frac{\partial G}{\partial r} = +\frac{1}{4\pi|\underline{x}-\underline{x}_0|^2}\frac{\partial|\underline{x}-\underline{x}_0|}{\partial r} - \frac{a}{r_0}\frac{1}{4\pi|\underline{x}-\underline{x}_1|^2}\frac{\partial}{\partial r}|\underline{x}-\underline{x}_1|$$

$$= \frac{1}{4\pi|\underline{x}-\underline{x}_0|^3}(a - r_0\cos\alpha) - \frac{a}{r_0 4\pi}\frac{r_0^2}{a^2|\underline{x}-\underline{x}_0|^2}\frac{r_0 - a\cos\alpha}{|\underline{x}-\underline{x}_0|}$$

$$= \frac{1}{4\pi|\underline{x}-\underline{x}_0|^3}\left\{a - r_0\cos\alpha - \frac{r_0}{a}(r_0 - a\cos\alpha)\right\}$$

$$= \frac{1}{4\pi a|\underline{x}-\underline{x}_0|^3}(a^2 - r_0^2)$$

Finally

$$u(\underline{x}_0) = \int_S U(\underline{x}) \frac{a^2 - r_0^2}{4\pi a |\underline{x} - \underline{x}_0|^3} \, dS$$

$$= \frac{a^2 - |\underline{x}_0|^2}{4\pi a} \int_S \frac{U(\underline{x})}{|\underline{x} - \underline{x}_0|^3} \, dS$$

A If G is chosen to satisfy the b.c. $G(\underline{x}, \underline{x}_0) = 0$ for $\underline{x} \in S$, then

$$u(\underline{x}_0) = \int_S U(\underline{x}) \frac{\partial G}{\partial n} \, dS$$

which is the solution for u
