2009 PART IIB 4M13 COMPLEX ANALYSIS AND OPTIMIZATION

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Section A, Complex analysis

1.

a)

i. Take out z^3 from the denominator to get

$$f(z)=\frac{1}{z^3(1-z^2)}=\frac{1}{z^3(1+z)(1-z)}$$

The singularities are all isolated, they are poles are at $z=-1,0,1$, of order 1, 3 and 1, respectively. Taking the 3rd order pole first, the Maclauren expansion of f around $z_0=0$ is

$$f(z)=\frac{1}{z^3}\big(1-z+z^2+O(z^3)\big)\big(1+z+z^2+O(z^3)\big)$$

The residue is the coefficient of the $1/z$ term, which is $2-1=1$.

For the first order poles, note that the residue at z_0 is $\lim_{z\to z_o}(z-z_0)f(z)$, so at $z_0=1$ we have

$$\lim_{z\to 1}\frac{(z-1)}{z^3(1+z)(1-z)} = -\frac{1}{2}$$

and at $z_0=-1$, we have

$$\lim_{z\to -1}\frac{z+1}{z^3(1+z)(1-z)} = -\frac{1}{2}$$

ii. The function $f(z)=1/\sin(1/z)$ has an essential singularity at $z=0$ and simple poles at the zeros of $\sin(1/z)$, which are at $z=1/n\pi$, $n=1,2...$ (there is also simple pole for $n=0$, in the extended complex plane that includes the point at infinity, but the course did not cover this). To obtain the residue at a simple pole, note that for z close to $1/n\pi$, $z=1/n\pi+\varepsilon$, where ε is small, and

$$\begin{aligned}\sin(1/z) &= \sin\left(\frac{1}{\frac{1}{n\pi}+\varepsilon}\right)\\ &= \sin\left(\frac{n\pi}{1+n\pi\varepsilon}\right)\\ &= \sin\big(n\pi(1-n\pi\varepsilon+O(\varepsilon^2))\big)\\ &= (-1)^n\sin(n^2\pi^2\varepsilon+O(\varepsilon^2))\\ &= (-1)^n n^2\pi^2\varepsilon+O(\varepsilon^2)\end{aligned}$$

using the periodicity of the sin function. Therefore the residue is

$(-1)^n/n^2\pi^2$

b) Use the substitution $z = e^{ix}$, $dz = izdx$ and integrate around the unit circle. Noting that $\cos x = (e^{ix} + e^{-ix}) = (z + 1/z)/2$ and that similarly $\sin x = (z - 1/z)/2i$, the complex integral is

$$I = \frac{1}{2^4} \oint \left(z + \frac{1}{z}\right)^4 + \left(z - \frac{1}{z}\right)^4 \frac{dz}{iz}.$$

The integrand has a pole at $z = 0$, and to get the coefficient of $1/z$ in the Maclauren expansion, note that

$$\left(z \pm \frac{1}{z}\right)^4 = \left(z^4 \pm 4\frac{z^3}{z} + 6\frac{z^2}{z^2} \pm 4\frac{z}{z^3} + \frac{1}{z^4}\right)$$

and that we need the constant terms from the parenthesized expressions, which together have a coefficient of 12, so using the residue theorem, the integral is

$$I = 2\pi i \frac{12}{16i} = \frac{3\pi}{2}$$

2.

a) Jordan's Lemma states that the integral of $g(z)e^{imz}$ where $m > 0$ and $g(z) \to 0$ as $|z| \to \infty$, along a semicircle of infinite radius in the upper half plane, is zero. If $m < 0$, complex conjugation implies that the integral vanishes along a similar semicircle the lower half plane.

b) The real function is even, so extend the integration over the whole of the real axis, and consider the complex version of the integral, using $z = x$,

$$\frac{1}{2}\int \frac{\cos mz}{z^2 + a^2} dz = \frac{1}{4}\int \frac{e^{imz} + e^{-imz}}{z^2 + a^2} dz$$

For the first term, complete the contour in the upper half plane, for the second term in the lower half plane, and the integral vanishes on the semicircles due to Jordan's Lemma. The two integrals are

$$\frac{1}{4}\oint \frac{e^{imz}}{z^2 + a^2} dz$$

and

$$\frac{1}{4}\oint \frac{e^{-imz}}{z^2 + a^2} dz$$

The poles of the integrand in both cases are at $z = \pm\, ia$. We no compute the residue of the first integrand at $z = ia$, the pole which is inside the contour.

$$\begin{aligned} \lim_{z \to ia} (z - ia) \times \frac{1}{4} \frac{e^{imz}}{(z + ia)(z - ia)} &= \frac{1}{4} \frac{e^{imia}}{(ia + ia)} \\ &= e^{-ma}/8ia \end{aligned}$$

The residue of the pole at $z = -\,ia$ is just the negative of this, but note that the second integral reverses sign because it goes around its residue clockwise, so the contributions of the two integrals are the same. Using the residue theorem, the integral is given by

$$2\pi i \times 2 \times \frac{e^{-ma}}{8ia} = \frac{\pi}{2a} e^{-ma}$$

Question 3

(a)

The volume of material used M is the volume of the two 'ends', $\pi(R+t)^2 t$ each, plus the volume of the cylinder $\pi\left[(R+t)^2 - R^2\right]H$. Hence

$$M = 2\pi(R+t)^2 t + \pi\left[(R+t)^2 - R^2\right]H$$

So the optimization problem is to minimize

$$f(R,H) = 2\pi(R+t)^2 t + \pi\left[(R+t)^2 - R^2\right]H$$

subject to $V = \pi R^2 H$ [10%]

(b)

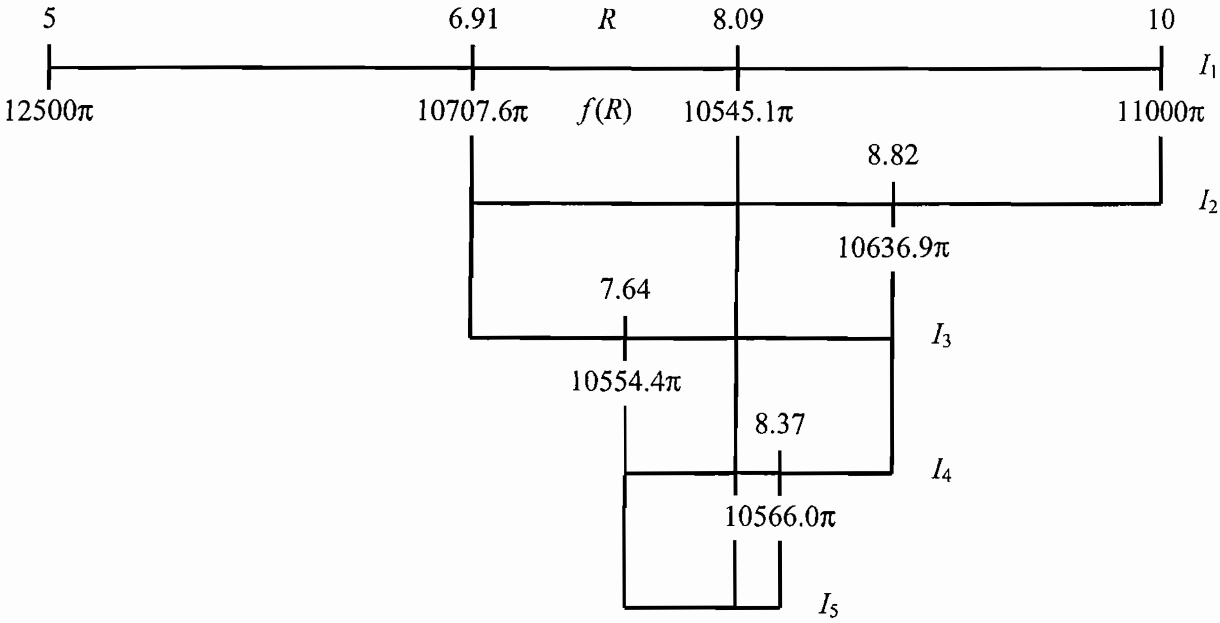
Substituting $H = \dfrac{V}{\pi R^2}$ $\quad f(R) = 2\pi(R+t)^2 t + \pi\left[(R+t)^2 - R^2\right]\dfrac{V}{\pi R^2}$

$$\therefore \quad f(R) = 2\pi(R+t)^2 t + V\left[\left(1+\frac{t}{R}\right)^2 - 1\right]$$ [10%]

(c)

For the values specified $\quad f(R) = 20\pi(R+10)^2 + 1000\pi\left[\left(1+\dfrac{10}{R}\right)^2 - 1\right]$

For the GSLS method $\dfrac{\Delta R}{I} = 0.382$



Hence $I_5 = 7.64 \text{ cm} \le R \le 8.37 \text{ cm}$ [40%]

(d)

$$f(R) = 2\pi(R+t)^2 t + V\left[\left(1+\frac{t}{R}\right)^2 - 1\right] = 2\pi(R+t)^2 t + V\left[\frac{2t}{R} + \frac{t^2}{R^2}\right]$$

$$\therefore \quad \frac{df}{dR} = 4\pi(R+t)t - V\left[\frac{2t}{R^2} + \frac{2t^2}{R^3}\right] = 4\pi(R+t)t - 2Vt\left[\frac{1}{R^2} + \frac{t}{R^3}\right] \qquad (1)$$

$$\therefore \qquad \frac{d^2 f}{dR^2} = 4\pi t + 2Vt\left[\frac{2}{R^3} + \frac{3t}{R^4}\right] \qquad (2)$$

For a minimum $\frac{df}{dR} = 0$ and $\frac{d^2 f}{dR^2} > 0$.

From (1) $$\frac{df}{dR} = 0 \quad \Rightarrow \quad 4\pi(R+t)t - 2Vt\left[\frac{1}{R^2} + \frac{t}{R^3}\right] = 0$$

$$\therefore \qquad 4\pi(R+t)t - \frac{2Vt}{R^3}[R+t] = 0$$

$$\therefore \qquad R^3 = \frac{V}{2\pi} \quad \Rightarrow \quad R = \sqrt[3]{\frac{V}{2\pi}} \qquad (3)$$

From (2) it is clear by inspection that $\frac{d^2 f}{dR^2} > 0$ for any positive value of R.

For the values given in part (c) $R = \sqrt[3]{\frac{1000\pi}{2\pi}} = \sqrt[3]{500} = 7.937$ cm

Hence $$H = \frac{V}{\pi R^2} = \frac{V}{\pi}\left(\frac{2\pi}{V}\right)^{\frac{2}{3}} = \sqrt[3]{\frac{4V}{\pi}} \qquad (4)$$

$$\therefore \qquad H = \sqrt[3]{\frac{4 \times 1000\pi}{\pi}} = \sqrt[3]{4000} = 15.874 \text{ cm}$$

This confirms that the GSLS is converging on the correct value. The order of convergence of GSLS is 1.618 (the golden ratio) – hence the method's name.

As (3) and (4) show, perhaps surprisingly, the optimal value of R does not depend on t. [40%]

Question 4

(a)

We want to maximize the surface area of the tubes

$$S = 2N\pi rL$$

where r is the radius of the tubes, N is the number of tubes and L is the length of the tubes (which is fixed in this case).

There is an upper limit on the cross-sectional area occupied by the tubes

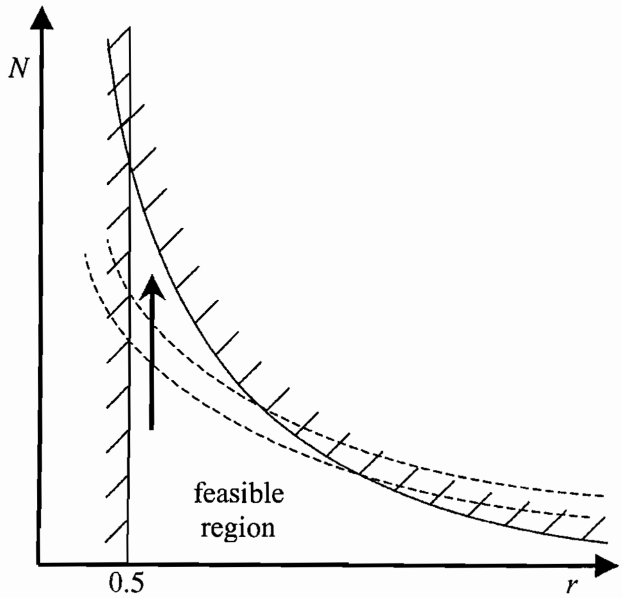
$$A = N\pi r^2 \leq 100 \text{ cm}^2$$

Hence, as a minimization problem, we have

$$\begin{aligned} &\text{Minimize} \quad f(N,r) = -2N\pi r \\ &\text{subject to} \quad N\pi r^2 \leq 100 \text{ cm}^2 \\ &\qquad\qquad\quad r \geq 0.5 \text{ cm} \end{aligned}$$

[10%]

(b)



The two constraint equations are shown as solid lines in the figure above (with infeasible space cross-hatched). Contours of the objective function are shown as dashed lines. The objective improves in the direction indicated by the arrow. Hence the optimum will be at the intersection of the two constraints (both constraints will be active).

Thus $r_{\text{opt}} = 0.5$ cm and $N_{\text{opt}} = \dfrac{100}{\pi r_{\text{opt}}^2} = \dfrac{100}{\pi \times 0.5^2} = 127.3$ (i.e. 127) [30%]

(c)

In standard K-TM form the problem is:

$$\begin{aligned} &\text{Minimize} \quad f(N,r) = -2N\pi r \\ &\text{subject to} \quad g_1 = N\pi r^2 - 100 \leq 0 \\ &\qquad\qquad\quad g_2 = 0.5 - r \leq 0 \end{aligned}$$

Hence $$L(N,r) = -2N\pi r + \mu_1\left(N\pi r^2 - 100\right) + \mu_2(0.5 - r)$$

$$\therefore \quad \frac{\partial L}{\partial N} = -2\pi r + \mu_1 \pi r^2 = 0 \qquad (1)$$

$$\frac{\partial L}{\partial r} = -2N\pi + \mu_1 2N\pi r - \mu_2 = 0 \qquad (2)$$

$$\mu_1\left(N\pi r^2 - 100\right) = 0 \qquad (3)$$

$$\mu_2\left(0.5 - r\ \right) = 0 \qquad (4)$$

Case 1: $\mu_1 = 0; \mu_2 = 0$

$(1) \Rightarrow -2\pi r = 0 \Rightarrow r = 0 \Rightarrow g_2$ is violated $\therefore$ impossible

Case 2: $\mu_1 = 0; \mu_2 > 0$

$(1) \Rightarrow -2\pi r = 0 \Rightarrow r = 0 \Rightarrow g_2$ is violated $\therefore$ impossible

Case 3: $\mu_1 > 0; \mu_2 = 0$

$(1) \Rightarrow -2\pi r + \mu_1 \pi r^2 = 0 \Rightarrow r = 0 \Rightarrow g_2$ is violated $\therefore$ impossible

$$\text{or} \Rightarrow \mu_1 = \frac{2}{r} \qquad (5)$$

$$(3) \Rightarrow N\pi r^2 - 100 = 0 \Rightarrow N = \frac{100}{\pi r^2} \qquad (6)$$

$(2) \Rightarrow -2N\pi + \mu_1 2N\pi r = 0 \Rightarrow N = 0$ which conflicts with (6)

or $\mu_1 = \dfrac{1}{r}$ which conflicts with (5)

$\therefore$ no solution for this case

Case 4: $\mu_1 > 0; \mu_2 > 0$

$(1) \Rightarrow -2\pi r + \mu_1 \pi r^2 = 0 \Rightarrow r = 0 \Rightarrow g_2$ is violated $\therefore$ impossible

$$\text{or} \Rightarrow \mu_1 = \frac{2}{r} \qquad (7)$$

$$(3) \Rightarrow N\pi r^2 - 100 = 0 \Rightarrow N = \frac{100}{\pi r^2} \qquad (8)$$

$(2) \Rightarrow -2N\pi + \mu_1 2N\pi r - \mu_2 = 0 \Rightarrow$

$$\mu_2 = 2N\pi\left(\mu_1 r - 1\right) = 2N\pi \text{ using (7)} \qquad (9)$$

Thus in Case 4 there is a solution:

From (4) $r_{opt} = 0.5$ cm

From (8) $N_{opt} = \dfrac{100}{\pi r_{opt}^2} = \dfrac{100}{\pi \times 0.5^2} = 127.3$ (i.e. 127)

From (7) $\mu_1 = \dfrac{2}{r} = \dfrac{2}{0.5} = 4$

From (9) $\mu_2 = 2N\pi = 254\pi$ [50%]

(d)

The values of the K-T multipliers indicate the sensitivity of the optimum to the constraints. The much higher value of μ_2 indicates that there will be a proportionately greater increase in the heat transfer area if smaller pipes can be used (reducing the limit on *r*) than if the cross-sectional area occupied by the tubes can be increased. [10%]