ENGINEERING TRIPOS PART IIB

Wednesday 6 May 2009 2.30 to 4

Module 4A8

ENVIRONMENTAL FLUID MECHANICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments: Data sheets (5 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 The geostrophic flow is governed by the following equation

$$2\underline{\Omega} \times \underline{u} = \frac{-1}{\rho_o} \nabla P,$$

where the symbols have their usual meaning. Take the rotation rate vector $\underline{\Omega} = (0, 0, \Omega_z)$ as a constant.

(a) What is the physical interpretation of this equation? Is the equation valid for planetary boundary layers? [15%]

(b) Show that the vertical component of the vorticity, $\underline{\omega} \equiv \nabla \times \underline{u}$, is related to the Laplacian of the pressure. Also, show that, when the pressure field is continuous, the geostrophic velocity field satisfies $\nabla \cdot \underline{u} = 0$. [35%]

(c) The air density ρ can be expressed as $\rho = \rho_o(1 - \alpha\, \theta)$, where α is the volumetric thermal expansion coefficient and $\theta = T(z) - T(0)$, with $T(z)$ the temperature at height z and $T(0)$ the ground temperature. A wind can result when the horizontal gradient of θ is sufficiently large. This wind is called *thermal wind* and is governed by

$$2\Omega_z \frac{\partial v}{\partial z} = g\alpha \frac{\partial \theta}{\partial x} \qquad \text{and} \qquad 2\Omega_z \frac{\partial u}{\partial z} = -g\alpha \frac{\partial \theta}{\partial y},$$

where u and v are the horizontal components of the velocity vector $\underline{u}$. Obtain the above two equations using the hydrostatic balance equation $\partial P / \partial z = -\rho g$ and the geostrophic equation given above. Determine the vertical variation of the thermal wind components u and v when the temperature increment varies as $\theta = Ax^2 + By$, where A and B are constants. Comment on your solutions. [50%]

2 Free convective flows in the environment can be approximated well by a flow between differentially heated parallel plates. Consider two horizontal infinitely long parallel plates separated by a vertical distance d. The bottom plate is at a temperature T_s and supplies a uniform heat flux of Q Wm^{-2}. After an initial transient, a stationary turbulent flow with zero mean velocity is created. Assume that horizontal variations are negligible. The mean density of the fluid in the gap is ρ and the mean specific heat capacity at constant pressure is c_p.

(a) Using the mean temperature equation from the Data Card, show that the turbulent heat flux in the vertical direction is given by

$$\overline{u_3\theta} \approx \frac{Q}{\rho c_p},$$

where the fluctuating velocity in the vertical direction is u_3, the temperature fluctuation is θ, and the mean molecular heat flux can be neglected. [20%]

(b) Simplify the equations for the turbulent kinetic energy k and for the variance σ of the temperature fluctuations (given in the Data Card) for this problem. Taking the body force in the balance equation for k as $g\,\overline{u_3\theta}/\overline{T}$, make an estimate for the turbulent kinetic energy per unit mass and its dissipation rate ε. The mean temperature is $\overline{T}$ and the gravitational constant is g. [30%]

(c) By taking the dissipation rate of σ as $c_d\,\sigma\,\varepsilon/k$, where c_d is a constant, show that the mean temperature at $d/2$ is

$$\frac{\overline{T}}{T_s} \approx (1-\mathscr{A}d)^{3/4},$$

where $\mathscr{A} = \dfrac{c_d}{3}\dfrac{\sigma}{T_s^{4/3}}\left(\dfrac{g}{d^2}\right)^{1/3}\left(\dfrac{\rho c_p}{Q}\right)^{2/3}$. [50%]

3 (a) Discuss briefly three of the major pollutants usually found in urban areas in terms of their sources, chemistry, health impacts, and diurnal variation. [70%]

(b) A lake of volume V_0 contains water contaminated by a pollutant at concentration ϕ_0 [units: kg m^{-3}]. At time $t = 0$, a water stream begins to flow into the lake at a constant volume flow rate Q carrying the same pollutant at concentration $2\phi_0$. Show that, if there is no outflow from the lake, the subsequent evolution of the pollutant concentration ϕ in the lake obeys

$$\frac{\phi}{\phi_0} = 2 - \left(1 + \frac{Q\,t}{V_0}\right)^{-1} .$$

[30%]

4 (a) Discuss briefly, using appropriate sketches, typical shapes of plumes from continuous sources in a uniform wind, including comments on the relevant atmospheric stability and inversions. [50%]

(b) Find the necessary relation between the dispersion coefficient σ and the diffusivity K for the one-dimensional Gaussian equation in the Data Card to be a solution to

$$U\frac{\partial \phi}{\partial x} = K\frac{\partial^2 \phi}{\partial y^2}$$

[50%]

END OF PAPER

4A8: Environmental Fluid Mechanics

Part I: Turbulence and Fluid Mechanics

DATA CARD

Rotating Flows

Geostrophic Flow

$$-\frac{1}{\rho}\nabla p = 2\underline{\Omega}\times\underline{u}$$

Ekman Layer Flow

$$-2\Omega_z v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}$$

$$2\Omega_z u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

OR

$$-2\Omega_z v = \nu\frac{\partial^2 u}{\partial z^2}$$

$$-2\Omega_z\left(u_g - u\right) = \nu\frac{\partial^2 v}{\partial z^2}$$

GEOSTROPHIC VELOCITY

Solution is

$$u = u_g\left[1 - e^{-z/\Delta}\cos\frac{z}{\Delta}\right]$$

$$v = u_g e^{-z/\Delta}\sin\frac{z}{\Delta}$$

$$\Delta = \left(\frac{\nu}{\Omega_z}\right)^{1/2}$$

Turbulent Flows – Incompressible

Continuity Equation $$\nabla \bullet \underline{U} = \frac{\partial U_i}{\partial x_i} = 0$$

Momentum Equation $$\rho \frac{DU_i}{Dt} = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 U_i}{\partial x_j^2} + F_i$$

Enthalpy Equation $$\rho c_p \frac{DT}{Dt} = -k \frac{\partial^2 T}{\partial x_i^2}$$

Reynolds Transformation $$U_i = \overline{U_i} + u_i \quad \text{etc}$$

Reynolds Stress $= -\rho \overline{u_i u_j}$, Reynolds Heat Flux $= -\rho \overline{c_p u_j \Theta}$

Turbulent Kinetic Energy, k, Equation

$$\frac{Dk}{Dt} = -\overline{u_i u_k} \frac{\partial \overline{U_i}}{\partial x_k} - \varepsilon + \frac{\overline{f_i u_i}}{\rho} + \text{ transport of kinetic energy forms}$$

Mean temperature equation

$$\rho c_p \frac{D\overline{T}}{Dt} = \frac{\partial}{\partial x_j}\left(\lambda \frac{\partial \overline{T}}{\partial x_j} - \overline{u_j \Theta} \rho c_p \right)$$

Temperature variance, σ, equation

$$\frac{D\sigma}{Dt} = -2\overline{u_j \Theta} \frac{\partial \overline{T}}{\partial x_j} - \varepsilon_\sigma + \text{molecular diffusion}$$

In flows with thermally driven motion

$$\frac{f_i u_i}{\rho} = \frac{g}{T} \bullet \overline{\Theta u_i}, \qquad i = \text{Vertical direction}$$

Dissipation of turbulent kinetic energy $$\varepsilon \approx \frac{u'^3}{\ell}$$

Kolmogorov microscale $$\eta = \left(\frac{\nu^3}{\varepsilon} \right)^{1/4}$$

Taylor microscale (λ) $$\varepsilon = 15\nu \frac{u'^2}{\lambda^2}$$ (ν is the kinematic viscocity)

Density Influenced Flows

Atmospheric Boundary Layer

$$\left.\frac{dT}{dz}\right|_{\text{NEUTRAL STABILITY}} = -\frac{g}{c_p} = \left.\frac{dT}{dz}\right|_{\text{DALR}}$$

$$R_i = \frac{g}{T}\frac{\frac{dT}{dz} - \left.\frac{dT}{dz}\right|_{\text{DALR}}}{\left(\frac{dU}{dz}\right)^2} = \text{RICHARDSON NUMBER}$$

Neutral Stability

$$U = \frac{u_*}{\kappa}\ln\frac{z}{z_0}; \quad \frac{dU}{dz} = \frac{u_*}{\kappa z}$$

$$u_* = \sqrt{\frac{\tau_w}{\rho}} \quad ; \quad \kappa = \text{Von Karman Constant} = 0.40$$

Non-Neutral Stability

$$\text{L} = \text{Monin-Obukhov length} = -\frac{u_*^3}{\kappa \frac{g}{T}\frac{Q}{\rho c_p}}$$

Q = surface heat flux

$$\frac{dU}{dz} = \frac{u_*}{\kappa z}\left(1 - 15\frac{z}{L}\right)^{-1/4} \quad \text{Unstable}$$

$$= \frac{u_*}{\kappa z}\left(1 + 4.7\frac{z}{L}\right) \quad \text{Stable}$$

Buoyant plume for a point source

$$\frac{d}{dz}\pi R^2 w = 2\pi R u_e \qquad \text{(i)}$$

$$\frac{d}{dz}\rho\pi R^2 w = \rho_a 2\pi R u_e \qquad \text{(ii)}$$

$$\frac{d}{dz}\rho\pi R^2 w^2 = g(\rho_a - \rho)\pi R^2 \qquad \text{(iii)}$$

(i) and (iii) give

$$\pi R^2 w\left(\frac{\rho_a - \rho}{\rho_a}\right)g = \text{constant} = F_0 \text{ (buoyancy flux)}$$

$$u_e = \alpha w$$

(α = Entrainment coefficient)

$$N^2 = -\frac{g}{\rho}\frac{d\rho}{dz} = \frac{g}{T}\frac{dT}{dz}$$

Actually $\frac{g}{T}\left(\frac{dT}{dz} - \left.\frac{dT}{dz}\right|_{\text{DALR}}\right)$

N = Brunt – Vaisala Frequency or Buoyancy Frequency

4A8: Environmental Fluid Mechanics

Part II: Dispersion of Pollution in the Atmospheric Environment

DATA CARD

Transport equation for the mean of the reactive scalar ϕ:

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_j \frac{\partial \overline{\phi}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(K \frac{\partial \overline{\phi}}{\partial x_j} \right) + \overline{w}$$

Transport equation for the variance of the reactive scalar ϕ:

$$\frac{\partial g}{\partial t} + \overline{u}_j \frac{\partial g}{\partial x_j} = \frac{\partial}{\partial x_j}\left(K \frac{\partial g}{\partial x_j} \right) + 2K\left(\frac{\partial \overline{\phi}}{\partial x_j} \right)^2 - \frac{2}{T_{turb}} g + 2\overline{\phi' w'}$$

Mean concentration of pollutant after instantaneous release of Q kg at t=0:

$$\overline{\phi}(x,y,z,t) = \frac{Q}{8(\pi t)^{3/2}(K_x K_y K_z)^{1/2}} \exp\left[-\frac{1}{4t}\left(\frac{(x-x_0)^2}{K_x} + \frac{(y-y_0)^2}{K_y} + \frac{(z-z_0)^2}{K_z} \right) \right]$$

Gaussian plume spreading in two dimensions from a source at (0,0,z_0) emitting Q kg/s:

$$\overline{\phi}(x,y,z) = \frac{Q}{2\pi} \frac{1}{U\sigma_y \sigma_z} \exp\left[-\left(\frac{y^2}{2\sigma_y^2} + \frac{(z-z_0)^2}{2\sigma_z^2} \right) \right]$$

One-dimensional spreading from line source emitting Q/L kg/s/m :

$$\overline{\phi}(x,y) = \frac{Q}{UL} \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left(-\frac{y^2}{2\sigma_y^2} \right)$$

Relationship between eddy diffusivity and dispersion coefficient:

$$\sigma^2 = 2\frac{x}{U}K$$