ENGINEERING TRIPOS PART IIB

Friday 1 May 2009 9 to 10.30

Module 4A12

TURBULENCE

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments: Data sheets (2 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 Consider a closed spherical vessel of radius R containing fluid of density ρ. An impeller is placed in the vessel to achieve stirring. Assume that the *mean* flow is zero everywhere and that the *mean* temperature of the fluid is homogeneous and constant.

(a) The impeller induces a typical large-scale turbulent velocity u_0 to the fluid. Assume that the turbulence is homogeneous and isotropic and that it is being maintained by the power P input to the impeller. Determine the turbulent lengthscale L_{turb}. [30%]

(b) The motor driving the impeller is now switched off. Assuming that L_{turb} does not change with time, determine the subsequent turbulent velocity scale u as a function of time. [40%]

(c) Before switching the motor off, the temperature fluctuations in the fluid have characteristic magnitude σ_0. Determine how the scalar fluctuations σ evolve in time after the motor is switched off. [30%]

2 The Reynolds stress equation for a steady incompressible turbulent flow is

$$\begin{aligned}\bar{u}_k \frac{\partial \overline{u'_i u'_j}}{\partial x_k} &= -\frac{1}{\rho}\left(\overline{u'_i \frac{\partial p'}{\partial x_j}} + \overline{u'_j \frac{\partial p'}{\partial x_i}}\right) - \frac{\partial \overline{u'_i u'_j u'_k}}{\partial x_k} + \frac{\mu}{\rho}\frac{\partial^2 \overline{u'_i u'_j}}{\partial x_k^2} \\ &\quad - \overline{u'_i u'_k}\frac{\partial \bar{u}_j}{\partial x_k} - \overline{u'_j u'_k}\frac{\partial \bar{u}_i}{\partial x_k} - 2\frac{\mu}{\rho}\overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_j}{\partial x_k}}\end{aligned}$$

(a) Sketch how the mean velocity and the individual components of the turbulent velocity fluctuations vary across a zero pressure gradient boundary layer. By reference to the Reynolds stress equation, discuss the reasons for any anisotropy present and the mechanism by which each component is generated. [50%]

(b) The governing equation for the mean flow in a zero pressure gradient boundary layer is:

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{\partial \overline{u'v'}}{\partial y} + \frac{\mu}{\rho}\frac{\partial^2 \bar{u}}{\partial y^2}$$

By using a reasonable model for the eddy viscosity and neglecting terms from the above equation as appropriate for the free turbulent layer, derive the log-law of the wall. [50%]

3 (a) Discuss the advantages of using two-equation turbulence models over simpler models. [50%]

(b) Explain the principal weaknesses of the standard $k - \varepsilon$ model. Give examples of flows in which each weakness becomes apparent. For each example you give, indicate how the limitation can be overcome. [50%]

4 The standard $k-\varepsilon$ model can be written as:

$$\frac{Dk}{Dt} = \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_j}\right) + P_k - \varepsilon$$

$$\frac{D\varepsilon}{Dt} = \frac{\partial}{\partial x_j}\left(\frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_j}\right) + c_{\varepsilon 1}\frac{\varepsilon}{k}P_k - c_{\varepsilon 2}\frac{\varepsilon^2}{k}$$

$$\nu_t = c_\mu \frac{k^2}{\varepsilon}$$

$$P_k = \frac{1}{2}\nu_t\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right)^2$$

$$c_\mu = 0.09,\ c_{\varepsilon 1} = 1.44,\ c_{\varepsilon 2} = 1.92,\ \sigma_k = 1.0,\ \sigma_\varepsilon = 1.3$$

(a) Derive a power-law expression for the decay rate of the turbulent kinetic energy k in homogeneous isotropic turbulence. Hence, or otherwise, deduce an expression for the energy dissipation rate ε as a function of time. [70%]

(b) Discuss briefly how you would set up in the laboratory homogeneous isotropic decaying turbulence and how you would measure k and ε. [30%]

END OF PAPER

Cambridge University Engineering Department

4A12: Turbulence

Data Card

Assume incompressible fluid with constant properties.

Continuity:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

Mean momentum:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \partial^2 \overline{u}_i / \partial x_j^2 - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \overline{g}_i$$

Mean scalar:

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_i \frac{\partial \overline{\phi}}{\partial x_i} = D\frac{\partial^2 \overline{\phi}}{\partial x_i^2} - \frac{\partial \overline{u_i' \phi'}}{\partial x_i}$$

Turbulent kinetic energy ($k = \overline{u_i' u_i'}/2$)**:**

$$\begin{aligned}\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} &= -\frac{1}{\rho}\frac{\partial \overline{u_j' p'}}{\partial x_j} - \frac{1}{2}\frac{\partial \overline{u_j' u_i' u_i'}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2} \\ &\quad - \overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{g_i' u_i'}\end{aligned}$$

Scalar fluctuations ($\sigma^2 = \overline{\phi' \phi'}$)**:**

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_j \frac{\partial \sigma^2}{\partial x_j} = D\frac{\partial^2 \sigma^2}{\partial x_j^2} - 2\overline{\phi' u_j' \frac{\partial \phi'}{\partial x_j}} - 2\overline{\phi' u_j'}\frac{\partial \overline{\phi}}{\partial x_j} - 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2}$$

Energy dissipation:

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \approx \frac{u^3}{L_{turb}}$$

Scalar dissipation:

$$2\overline{N} = 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2} \approx 2\frac{\varepsilon}{k}\sigma^2$$

Scaling rule for shear flow, flow dominant in direction x_1:

$$\frac{u}{L_{turb}} \sim \frac{\partial \overline{u}_1}{\partial x_2}$$

Kolmogorov scales:

$$\begin{aligned} \eta_K &= \left(\nu^3/\varepsilon\right)^{1/4} \\ \tau_K &= \left(\nu/\varepsilon\right)^{1/2} \\ v_K &= \left(\nu\varepsilon\right)^{1/4} \end{aligned}$$

Taylor microscale:

$$\varepsilon = 15\nu\frac{u^2}{\lambda^2}$$

Eddy viscosity (general):

$$\begin{aligned} \overline{u_i' u_j'} &= -\nu_{turb}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \frac{2}{3}k\delta_{ij} \\ \overline{u_j' \phi'} &= -D_{turb}\frac{\partial \overline{\phi}}{\partial x_j} \end{aligned}$$

Eddy viscosity (for simple shear):

$$\overline{u_1' u_2'} = -\nu_{turb}\frac{\partial \overline{u}_1}{\partial x_2}$$