ENGINEERING TRIPOS PART IIB

Tuesday 21 April 2009 2.30 to 4

Module 4C2

DESIGNING WITH COMPOSITES

Answer not more than **three** *questions.*

All questions carry the same number of marks.

The **approximate** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
4C2 Designing with Composites datasheet (6 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) Show that the Young's modulus of a unidirectional composite (having the elastic constants given below) falls by about 50% if it is loaded at 7° to the fibre axis, compared with the on-axis value. What is the minimum Young's modulus that the lamina can exhibit and at what loading angle does this occur?
[E_1 = 200 GPa, E_2 = 7 GPa, G_{12} = 3 GPa, ν_{12} = 0.3] [35%]

(b) Explain why a *laminate* is more commonly employed than an unidirectionally reinforced composite material. Define a *quasi-isotropic* laminate, and write down the form of the [A] matrix. [35%]

(c) A simple cross-ply laminate is made from two unidirectional plies of a composite material. Each ply contains a volume fraction f = 0.66 of glass fibres of Young's modulus E_f = 76 GPa in a polyester matrix of Young's modulus E_m = 3 GPa. The laminate is subjected to uniaxial tensile loading.

(i) Determine the axial Young's modulus of each ply assuming an equal strain model, and determine the transverse Young's modulus of each ply assuming an equal stress model. [10%]

(ii) The cross-ply laminate is subjected to a tensile stress of 100 MPa aligned with the fibre direction in one of the plies. Using the values from (i) above, calculate the axial strain exhibited by the laminate, and sketch its distortion out-of-plane. How could the cross-ply laminate be constructed to eliminate this shape change? [20%]

2 (a) Comment on the following observations.

(i) Joints are often a source of failure in composites. [20%]

(ii) The resin system to be used in a composite aircraft structure cannot easily be changed late in the aircraft design process. [20%]

(b) Describe the vacuum injection moulding process and explain why it is used for boat hulls. [30%]

(c) Outline the shear lag model as applied to the longitudinal tensile failure of composites and explain how the model can help understand the details of this failure mode. [30%]

3 (a) Comment on the following design rules of thumb. Why are they often proposed and how useful are they?

(i) simplify
(ii) use a balanced symmetric laminate
(iii) use the same material in all plies
(iv) have at least 10% off-axis plies
(v) use either $\pm\theta$ or a sequence of $0°$, $90°$ and $\pm 45°$ plies [25%]

(b) A composite cantilever beam of length ℓ and width w has a sandwich structure with core thickness c and face plate thickness t, as illustrated in Fig. 1. A load F is applied at the tip of the cantilever.

(i) Assuming that $t \ll c$ so that bending moments in the face plates can be neglected, show that the maximum strain in the face plate is

$$\varepsilon_{\max} = \frac{F\ell}{cwtE_x}$$

where E_x is the face plate modulus along the axis of the beam. End effects and the stiffness of the core can be neglected. [15%]

(ii) The face plates are to be made from CFRP, GFRP or Kevlar. The core thickness of the beam is fixed but the thickness of the face plates can be varied depending on the material. There is a premium of £20/kg for reduction in the weight of the beam. Use Table 1 in the data sheet to select which of the three composites should be chosen to maximise the profit for making the beam, while carrying the given tip load without failing. [30%]

(iii) The applied load and other beam properties have been fixed at $F = 100$ N, $\ell = 3$ m, $w = 0.1$ m and $c = 0.02$ m. In addition each of the beam face plates suffer an in-plane shear line load $N_{xy} = 50$ kN/m. The face plates are identical and comprise $0°$, $90°$ and $\pm 45°$ plies of GFRP composite. Suggest an appropriate thickness for each ply direction in the face plates. [30%]

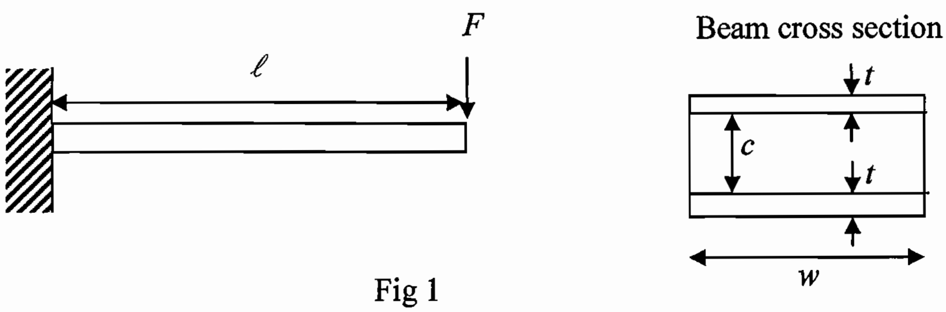


Fig 1

4 (a) Mechanical testing under 'hot-wet' environmental conditions is considered critical in the testing of aerospace materials. What mechanical properties are likely to be most affected by this environment, explaining your reasoning? [20%]

(b) Fig. 2 shows a +20° unidirectional ply made of Kevlar 49/934 composite (material properties on the data sheet). Calculate the value of the applied stress σ_x at failure, using the Tsai-Hill failure criterion. [25%]

(c) A $[\pm 20°]_S$ laminate made of four plies of Kevlar 49/934 of equal thickness is loaded by a stress σ_x.

(i) Explain, by referring to the ply stresses developed in each ply, why the failure stress of the laminate differs from that of the lamina of part (b). [10%]

(ii) Calculate the value of σ_x at first-ply failure for the laminate, now using the maximum strain failure criterion. The $[\bar{Q}]$ matrix for a 20° lamina is as follows:

$$[\bar{Q}_{20}] = \begin{bmatrix} 60 & 9 & 20 \\ 9 & 7 & 3 \\ 20 & 3 & 9 \end{bmatrix} \text{GPa}.$$ [45%]

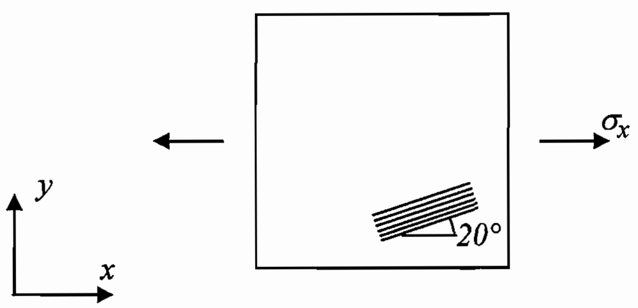


Fig 2

END OF PAPER

ENGINEERING TRIPOS PART II B

Module 4C2 – Designing with Composites

DATA SHEET

The in-plane compliance matrix [S] for a transversely isotropic lamina is defined by

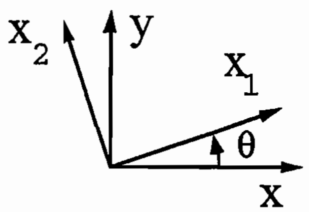
$$\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix}=[S]\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_{12}\end{pmatrix}\qquad \text{where}\,[S]=\begin{bmatrix}1/E_1 & -\nu_{21}/E_2 & 0\\ -\nu_{12}/E_1 & 1/E_2 & 0\\ 0 & 0 & 1/G_{12}\end{bmatrix}$$

[S] is symmetric, giving $\nu_{12}/E_1=\nu_{21}/E_2$. The compliance relation can be inverted to give

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_{12}\end{pmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix}\qquad \begin{aligned}\text{where}\ Q_{11} &= E_1/(1-\nu_{12}\nu_{21})\\ Q_{22} &= E_2/(1-\nu_{12}\nu_{21})\\ Q_{12} &= \nu_{12}E_2/(1-\nu_{12}\nu_{21})\\ Q_{66} &= G_{12}\end{aligned}$$

Rotation of co-ordinates

Assume the principal material directions (x_1,x_2) are rotated anti-clockwise by an angle θ, with respect to the (x,y) axes.



Then, $$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_{12}\end{pmatrix}=[T]\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{pmatrix}$$ and $$\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix}=[T]^{-T}\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{pmatrix}$$

where $$[\mathrm{T}]=\begin{bmatrix}\cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta\\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta\\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta-\sin^2\theta\end{bmatrix}$$

and $$[\mathrm{T}]^{-T}=\begin{bmatrix}\cos^2\theta & \sin^2\theta & \sin\theta\cos\theta\\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta\\ -2\sin\theta\cos\theta & 2\sin\theta\cos\theta & (\cos^2\theta-\sin^2\theta)\end{bmatrix}$$

The stiffness matrix $[Q]$ transforms in a related manner to the matrix $[\bar{Q}]$ when the axes are rotated from (x_1,x_2) to (x,y)

$$[\bar{Q}]=[T]^{-1}[Q][T]^{-T}$$

In component form,

$$[\bar{Q}]=\begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix} \text{ where } \begin{aligned}
\bar{Q}_{11} &= Q_{11}c^4+Q_{22}s^4+2(Q_{12}+2Q_{66})s^2c^2\\
\bar{Q}_{12} &= (Q_{11}+Q_{22}-4Q_{66})s^2c^2+Q_{12}\left(c^4+s^4\right)\\
\bar{Q}_{22} &= Q_{11}s^4+Q_{22}c^4+2(Q_{12}+2Q_{66})s^2c^2\\
\bar{Q}_{16} &= (Q_{11}-Q_{12}-2Q_{66})sc^3-(Q_{22}-Q_{12}-2Q_{66})s^3c\\
\bar{Q}_{26} &= (Q_{11}-Q_{12}-2Q_{66})s^3c-(Q_{22}-Q_{12}-2Q_{66})sc^3\\
\bar{Q}_{66} &= (Q_{11}+Q_{22}-2Q_{12}-2Q_{66})s^2c^2+Q_{66}\left(s^4+c^4\right)
\end{aligned}$$

with $c=\cos\theta,\quad s=\sin\theta$

The compliance matrix $[S]\equiv[Q]^{-1}$ transforms to $[\bar{S}]\equiv[\bar{Q}]^{-1}$ under a rotation of co-ordinates by θ from (x_1,x_2) to (x,y), as

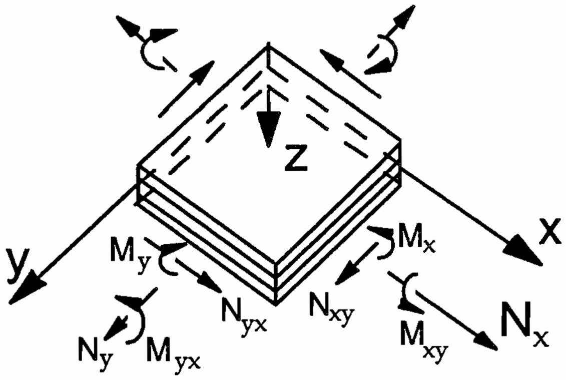
$$[\bar{S}]=[T]^T[S][T]$$

and in component form,

$$\begin{aligned}
\bar{S}_{11} &= S_{11}c^4+S_{22}s^4+(2S_{12}+S_{66})s^2c^2\\
\bar{S}_{12} &= S_{12}\left(c^4+s^4\right)+(S_{11}+S_{22}-S_{66})s^2c^2\\
\bar{S}_{22} &= S_{11}s^4+S_{22}c^4+(2S_{12}+S_{66})s^2c^2\\
\bar{S}_{16} &= (2S_{11}-2S_{12}-S_{66})sc^3-(2S_{22}-2S_{12}-S_{66})s^3c\\
\bar{S}_{26} &= (2S_{11}-2S_{12}-S_{66})s^3c-(2S_{22}-2S_{12}-S_{66})sc^3\\
\bar{S}_{66} &= (4S_{11}+4S_{22}-8S_{12}-2S_{66})s^2c^2+S_{66}\left(c^4+s^4\right)
\end{aligned}$$

with $c=\cos\theta,\quad s=\sin\theta$

Laminate Plate Theory



Consider a plate subjected to stretching of the mid-plane by $(\varepsilon_x^o, \varepsilon_y^o, \varepsilon_{xy}^o)^T$ and to a curvature $(\kappa_x, \kappa_y, \kappa_{xy})^T$. The stress resultants $(N_x, N_y, N_{xy})^T$ and bending moment per unit length $(M_x, M_y, M_{xy})^T$ are given by

$$\begin{pmatrix} N \\ \cdots \\ M \end{pmatrix} = \begin{bmatrix} A & \vdots & B \\ \cdots & . & \cdots \\ B & \vdots & D \end{bmatrix} \begin{pmatrix} \varepsilon^o \\ \cdots \\ \kappa \end{pmatrix}$$

In component form, we have,

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}$$

where the laminate extensional stiffness, A_{ij}, is given by:

$$A_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k dz = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k - z_{k-1}\right)$$

the laminate coupling stiffnesses is given by

$$B_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k z dz = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k^2 - z_{k-1}^2\right)$$

and the laminate bending stiffness are given by:

$$D_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k z^2 dz = \frac{1}{3}\sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k^3 - z_{k-1}^3\right)$$

with the subscripts $i, j = 1, 2$ or 6.

Here,

$n =$ number of laminae

$t =$ laminate thickness

$z_{k\text{-}1} =$ distance from middle surface to the top surface of the *k-th* lamina

$z_k =$ distance from middle surface to the bottom surface of the *k-th* lamina

Quadratic failure criteria.

For plane stress with $\sigma_3 = 0$, failure is predicted when

Tsai-Hill: $$\frac{\sigma_1^2}{s_L^2} - \frac{\sigma_1\sigma_2}{s_L^2} + \frac{\sigma_2^2}{s_T^2} + \frac{\tau_{12}^2}{s_{LT}^2} \geq 1$$

Tsai-Wu: $$F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + F_1\sigma_1 + F_2\sigma_2 + 2F_{12}\sigma_1\sigma_2 \geq 1$$

where $F_{11} = \frac{1}{s_L^+ s_L^-}$, $F_{22} = \frac{1}{s_T^+ s_T^-}$, $F_1 = \frac{1}{s_L^+} - \frac{1}{s_L^-}$, $F_2 = \frac{1}{s_T^+} - \frac{1}{s_T^-}$, $F_{66} = \frac{1}{s_{LT}^2}$

F_{12} should ideally be optimised using appropriate strength data. In the absence of such data, a default value which should be used is

$$F_{12} = -\frac{(F_{11}F_{22})^{1/2}}{2}$$

Fracture mechanics

Consider an orthotropic solid with principal material directions x_1 and x_2. Define two effective elastic moduli E'_A and E'_B as

$$\frac{1}{E'_A}=\left(\frac{S_{11}S_{22}}{2}\right)^{1/2}\left(\left(\frac{S_{22}}{S_{11}}\right)^{1/2}\left(1+\frac{2S_{12}+S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

$$\frac{1}{E'_B}=\left(\frac{S_{11}S_{22}}{2}\right)^{1/2}\left(\left(\frac{S_{11}}{S_{22}}\right)^{1/2}\left(1+\frac{2S_{12}+S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

where S_{11} etc. are the compliances.

Then G and K are related for plane stress conditions by:

crack running in x_1 direction: $G_I E'_A = K_I^2; G_{II} E'_B = K_{II}^2$

crack running in x_2 direction: $G_I E'_B = K_I^2; G_{II} E'_A = K_{II}^2$.

For mixed mode problems, the total strain energy release rate G is given by

$G = G_I + G_{II}$

Approximate design data

	Steel	Aluminium	CFRP	GFRP	Kevlar
Cost C (£/kg)	1	2	100	5	25
E_1 (GPa)	210	70	140	45	80
G (GPa)	80	26	≈35	≈11	≈20
ρ (kg/m^3)	7800	2700	1500	1900	1400
e^+ (%)	0.1-0.8	0.1-0.8	0.4	0.3	0.5
e^- (%)	0.1-0.8	0.1-0.8	0.5	0.7	0.1
e_{LT} (%)	0.15-1	0.15-1	0.5	0.5	0.3

Table 1. Material data for preliminary or conceptual design. Costs are very approximate.

	Aluminium	Carbon/epoxy (AS/3501)	Kevlar/epoxy (Kevlar 49/934)	E-glass/epoxy (Scotchply/1002)
Cost (£/kg)	2	100	25	5
Density (kg/m^3)	2700	1500	1400	1900
E_1 (GPa)	70	138	76	39
E_2 (GPa)	70	9.0	5.5	8.3
ν_{12}	0.33	0.3	0.34	0.26
G_{12} (GPa)	26	6.9	2.3	4.1
s_L^+ (MPa)	300 (yield)	1448	1379	1103
s_L^- (MPa)	300	1172	276	621
s_T^+ (MPa)	300	48.3	27.6	27.6
s_T^- (MPa)	300	248	64.8	138
s_{LT} (MPa)	300	62.1	60.0	82.7

Table 2. Material data for detailed design calculations. Costs are very approximate.

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ENGINEERING TRIPOS PART IIB 2009

MODULE 4C2: DESIGNING WITH COMPOSITES

NUMERICAL ANSWERS

Q1. (a) E= 102.9 GPa. At angle of 69.8 degrees, E=6.87 GPa.
(c) (i) Axial E=51.2 GPa, Transverse E=8.2 GPa, for cross-ply E=29.7 GPa
(c) (ii) strain = 0.0034.

Q3. (b) (ii) choose CFRP (iii) 1.1mm for 0 plies, 0.91mm for +-45 plies, 0.2mm for 90 plies.

Q4. (b) 146 MPa. (c) (ii) 570 MPa.

Norman A Fleck