ENGINEERING TRIPOS PART IIB

Monday 20 April 2009 9 to 10.30

Module 4C7

RANDOM AND NON-LINEAR VIBRATIONS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

Candidates may bring their notebooks to the examination.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
CUED approved calculator allowed
Engineering Data Book

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 An oil production vessel is subjected to a storm in which the *single sided* spectrum $S_{\eta\eta}(\omega)$ of the surface elevation of the ocean $\eta(t)$ has the form shown in Fig. 1(a). The motion of the vessel is described in terms of the heave $x(t)$ and pitch $\theta(t)$ at the centre of gravity, as shown in Fig. 1(b). Oil processing equipment is located at a distance b forward of the centre of gravity, and the vertical displacement at this point is labelled $y(t)$. The transfer functions between the heave and pitch motions and the surface elevation are given by

$$\text{Heave: } H_{x\eta}(\omega) = \alpha\omega\,,$$

$$\text{Pitch: } H_{\theta\eta}(\omega) = i\beta / \omega\,,$$

where α and β are constants.

(a) Assuming that the pitch rotation is small, write down an expression for $y(t)$ in terms of $x(t)$ and $\theta(t)$, and hence find the transfer function between $y(t)$ and the surface elevation. [10%]

(b) Find an expression for the mean squared value of the acceleration $a(t) = \ddot{y}(t)$ and also for the mean squared value of the rate of change of the acceleration, $\dot{a}(t) = \dddot{y}(t)$. [40%]

(c) Calculate the probability that the acceleration $a(t)$ will exceed 2.5 m s^{-2} at least once during a three hour storm for the particular case: $b = 50$ m, $\omega_1 = 0.3 \text{ rad s}^{-1}$, $\omega_2 = 0.8 \text{ rad s}^{-1}$, $S_0 = 30 \text{ m}^2 \text{ s rad}^{-1}$, $\beta = 4\times10^{-3} \text{ rad}^2 \text{ m}^{-1} \text{ s}^{-1}$ and $\alpha = 0.4 \text{ rad}^{-1}\text{s}$. [50%]

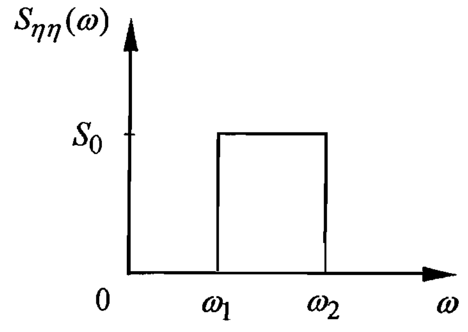

Fig. 1(a)

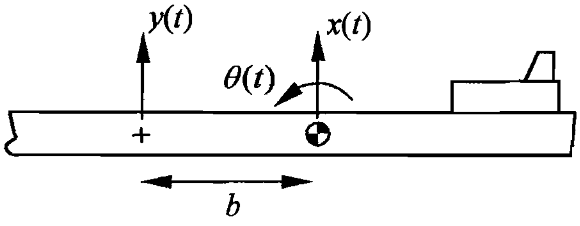

Fig. 1(b)

2 A very approximate model of an aircraft undercarriage is shown in Fig. 2. The aircraft is represented by a fixed boundary, and the undercarriage stiffness and damping constants are denoted by k_1 and C respectively. The tyre stiffness is denoted by k_2 and the mass of the undercarriage is M. The aircraft is driven over rough ground, and this produces a prescribed motion $y(t)$ at the base of the tyre, as shown in Fig. 2. The motion of the undercarriage (i.e. the motion of the mass M) in response to this input is denoted by $x(t)$.

(a) Derive an expression for the spectrum $S_{xx}(\omega)$ of the undercarriage response in terms of the spectrum $S_{yy}(\omega)$ of the ground input. [25%]

(b) For the case in which the ground input spectrum is white noise with $S_{yy}(\omega) = S_0$, derive expressions for the rms displacement σ_x and velocity $\sigma_{\dot{x}}$ of the undercarriage. The ground input spectrum should be considered to be double-sided (i.e. defined for both positive and negative frequencies). [25%]

(c) Explain in physical terms why $\sigma_{\dot{x}}$ is independent of k_1. [15%]

(d) Calculate the ensemble average of the power dissipated by the damper, and show that this is independent of both k_1 and C. Explain this result in physical terms, and discuss whether the result would also hold for a non-white input $S_{yy}(\omega)$. [35%]

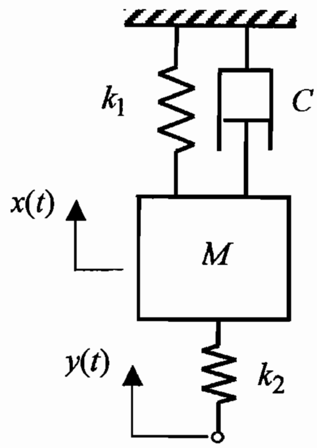

Fig. 2

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3 An undamped Duffing oscillator has the equation of motion

$$\ddot{x} + \alpha x + \beta x^3 = 0,$$

where α and β are constants.

(a) Plot the behaviour of the system in phase space for the case $\beta > 0$. [20%]

(b) Sketch the bifurcation diagram of the system with respect to the parameter α. What type of bifurcation does the system exhibit? [20%]

(c) The oscillator is now subjected to forcing at two input frequencies, so that the equation of motion becomes

$$\ddot{x} + \alpha x + \beta x^3 = F_1 \cos \Omega_1 t + F_2 \cos \Omega_2 t.$$

Solve for the response of the system to first order by using the method of iteration. You may assume that $\beta > 0$ and $\alpha \gg \beta$. Demonstrate that the response is comprised of combinations of the forcing frequencies, and list these combinations. [60%]

4 A vibratory system with non-linear damping has the equation of motion

$$\ddot{x} + \varepsilon \sin \dot{x} + x = 0,$$

where $x(t)$ is the displacement and $0 < \varepsilon < 1$.

(a) Transform the equation of motion to two first order differential equations and determine the singular or equilibrium points of the system. [20%]

(b) Comment on the type and stability of each equilibrium point. [20%]

(c) Sketch the behaviour of the system in the phase plane. Demonstrate qualitatively that the system can describe an infinite set of limit cycles and comment on the stability of these limit cycles. [50%]

(d) Describe how the phase portrait will change if $-1 < \varepsilon < 0$. [10%]

END OF PAPER

4C7 2009 Answers

1. (a) $H_{y\eta}(\omega) = \alpha\omega - ib\beta / \omega$

(b) $\sigma_{\dot{y}}^2 = (S_0\alpha^2 / 7)(\omega_2^7 - \omega_1^7) + (S_0 b^2 \beta^2 / 3)(\omega_2^3 - \omega_1^3)$

$\sigma_{\ddot{y}}^2 = (S_0\alpha^2 / 9)(\omega_2^9 - \omega_1^9) + (S_0 b^2 \beta^2 / 5)(\omega_2^5 - \omega_1^5)$

(c) $P \approx 0.1$

2. (a) $S_{xx}(\omega) = \dfrac{(k_2 / M)^2 S_{yy}(\omega)}{(\omega_n^2 - \omega^2)^2 + (2\beta\omega_n\omega)^2}$, $\omega_n^2 = (k_1 + k_2)/M$, $2\beta\omega_n = C/M$.

(b) $\sigma_x^2 = \dfrac{\pi k_2^2 S_0}{C(k_1 + k_2)}$, $\sigma_{\dot{x}}^2 = \dfrac{\pi k_2^2 S_0}{MC}$.

(d) $\mathrm{E}[P] = \dfrac{\pi k_2^2 S_0}{M}$

3. (b) There is a pitchfork bifurcation.

(c) The response has frequency components:
$\Omega_1, \Omega_2, 2\Omega_1 \pm \Omega_2, 2\Omega_2 \pm \Omega_1, 3\Omega_1, 3\Omega_2$.

4. (a),(b) Stable spiral at $x = \dot{x} = 0$.

(c) For $0 < \dot{x} < \pi$ the damping is positive and the amplitude decays; for $\pi < \dot{x} < 2\pi$ the damping is negative and the amplitude grows; for $2\pi < \dot{x} < 3\pi$ the damping is positive and the amplitude decays, etc. This gives an unstable limit cycle at radius π around the origin; a stable limit cycle at 2π around the origin, etc.