ENGINEERING TRIPOS PART IIB

Friday 24 April 2009 2.30 to 4

Module 4C8

APPLICATIONS OF DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
4C8 datasheet (4 pages)

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) Figure 1 shows a railway wheelset with effective conicity ε, average wheel radius r and track gauge $2d$, moving along a horizontal track with radius of curvature R at steady speed u. It has small lateral tracking error y and small yaw angle θ. The coefficients of *both* lateral and longitudinal creep of the wheels are C.

Show that the net lateral force Y and net yaw moment N acting on the wheelset due to the creep forces are given by:

$$Y = 2C\left(\theta + \frac{\dot{y}}{u}\right)$$

$$N = 2dC\left(\frac{\varepsilon y}{r} - \frac{d\dot{\theta}}{u} - \frac{d}{R}\right)$$

and indicate their directions on a sketch of the wheelset. State your assumptions. [40%]

(b) Figure 2 shows a bogie, comprising two such wheelsets, connected together by a rigid frame at a spacing of $2a$.

(i) Derive an equation for the motion of the lateral tracking error y of the centre of the bogie, incorporating the radius of curvature of the track. [20%]

(ii) Find an expression for the wavelength of the hunting motion on straight track. Compare it with the hunting wavelength of a free wheelset. [10%]

(iii) The bogie runs along a track which has a lateral displacement that varies sinusoidally with a wavelength L that is double the hunting wavelength in (ii), and has amplitude Δ. What is the amplitude of the lateral tracking error? [30%]

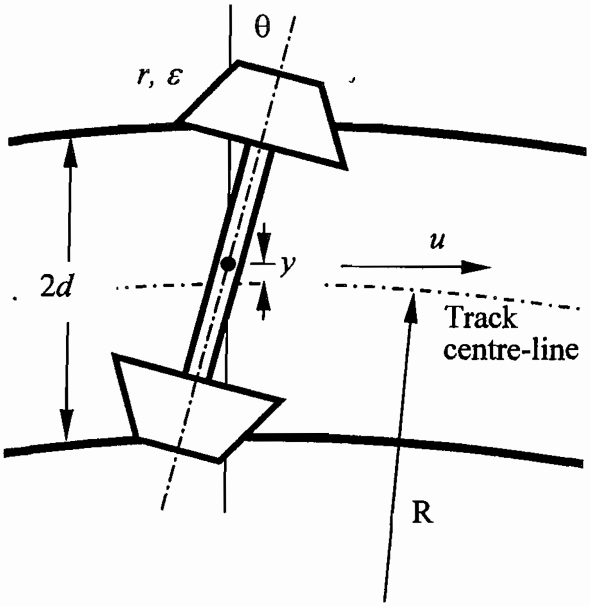

Fig. 1

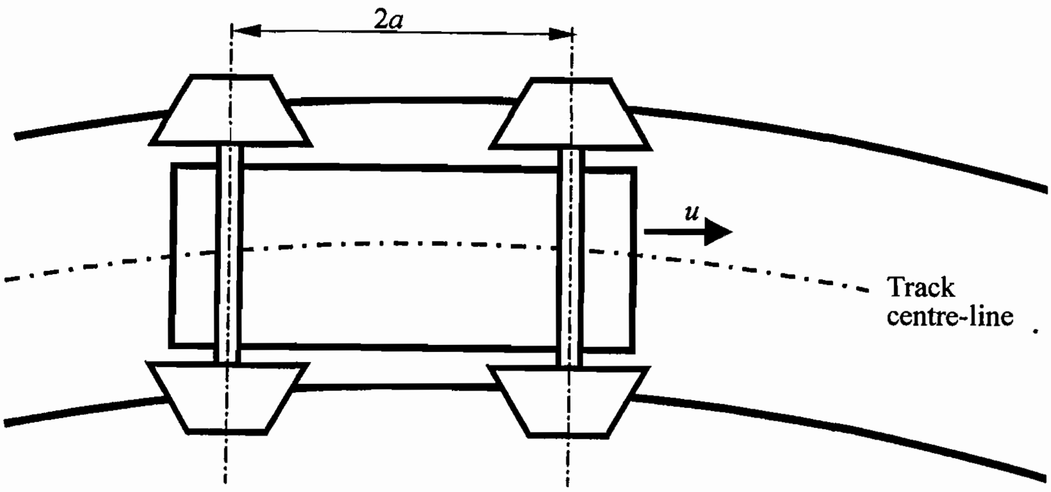

Fig. 2

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2 (a) The castered wheel shown in Fig. 3 is towed behind a moving vehicle by a frictionless joint at P. The distance from P to the contact point Q is a. The moment of inertia of the caster and wheel assembly about P is I, and the coefficient of lateral creep of the wheel is C. Point P moves with steady forward speed u and no lateral velocity. At time $t = 0$ point P is given an additional speed component of $\dot{y} = v$ in the lateral direction, with $v << u$. Sketch a graph of the variation of the caster angle θ (assumed small) as a function of time, showing salient values.

Use the following data: $a = 0.05$m; $I = 5\text{x}10^{-4}$ kg m^2; $v = 0.1$ m/s; $u = 1$ m/s; and $C = 40$ N/rad. [50%]

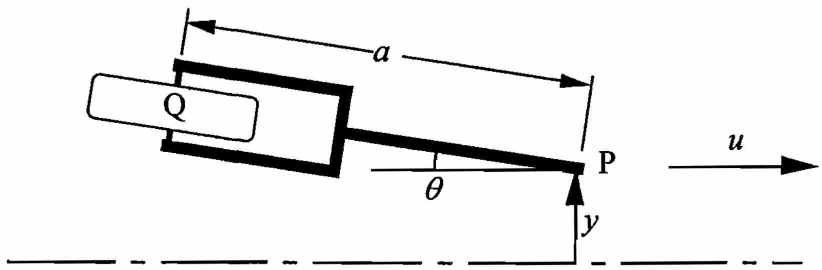

Fig. 3

(b) The contact between a pneumatic tyre and a road surface can be modelled as a 'brush' with a circular contact area of radius a and stiffness per unit area of K in the lateral direction. Assuming no microslip in the contact area (i.e. full friction), calculate the lateral force Y when the tyre rolls with a small yaw angle δ to the rolling direction. State your assumptions. [50%]

3 (a) Show, with the aid of diagrams if necessary, that the external gravitational potential of a uniform sphere with mass M is the same as that of a point mass M located at the centre of the sphere. Would this result be changed if the density of the sphere varied with radius? [40%]

(b) (i) A body of mass m moves in a closed Keplerian orbit under the gravitational attraction of a fixed body of mass M. Using the equations in sections 4 and 5 of the data sheet, derive an equation for the shape of the orbit. [30%]

(ii) By considering the periapsis of the orbit, show that the total energy E of the orbiting body is given by:

$$E = \frac{mM^2G^2}{2h^2}\left(e^2 - 1\right)$$

where e is the eccentricity of the orbit, and h is the angular momentum per unit mass. For what range of values of e is the orbit closed? And for a given value of h, what orbit shape corresponds to the lowest value of E? [30%]

(TURN OVER

4 (a) Explain the physical origin of the J_2 term in the expression for the external potential of the Earth given on the data sheet. [15%]

(b) A thin ring of matter, with radius R and total mass m, is centred at the origin of a system of spherical polar co-ordinates, with its axis of symmetry coinciding with the polar axis, as shown in Fig. 4. Show that the gravitational potential at point P, whose co-ordinates are $(r, \theta, 0)$ can be written as:

$$U = \frac{mG}{2\pi}\int_0^{2\pi}\frac{d\phi}{\sqrt{r^2 + R^2 - 2rR\sin\theta\cos\phi}}$$ [30%]

(c) For the case where $r \gg R$, use the binomial theorem to evaluate this integral approximately, keeping terms up to the order R^2 / r^2. [35%]

(d) The Earth can be modelled as a uniform sphere of mass M and radius R plus an additional ring of the type investigated in part (b) around the equator, such that $M_{Earth} = M + m$. Use the expression derived in part (c), and the value of J_2 in the data sheet, to determine what fraction of the Earth's total mass should be modelled into the ring. [20%]

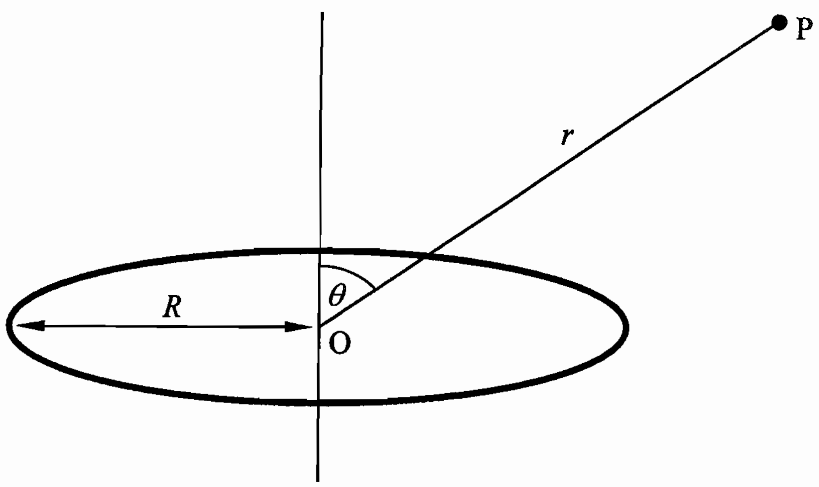

Fig. 4

END OF PAPER

Engineering Tripos Part IIB

Data sheet for Module 4C8: Applications of Dynamics

DATA ON VEHICLE DYNAMICS

1. Creep Forces In Rolling Contact

1.1 Surface tractors

Longitudinal force $X = \iint_A \sigma_x \, dA$

Lateral force $Y = \iint_A \sigma_y \, dA$

Realigning Moment $N = \iint_A (x\,\sigma_y - y\,\sigma_x)\,dA$

where

σ_x, σ_y = longitudinal, lateral surface tractions

x, y = coordinates along, across contact patch

A = area of contact patch

1.2 Brush model

$\sigma_x = K_x q_x$, $\sigma_y = K_y q_y$ for $\sqrt{\sigma_x^2 + \sigma_y^2} \le \mu p$

where

q_x, q_y = longitudinal, lateral displacements of 'bristles' relative to wheel rim

K_x, K_y = longitudinal, lateral stiffness per unit area

μ = coefficient of friction

p = local contact pressure

1.3 Linear creep equations

$X = -C_{11}\xi$

$Y = -C_{22}\alpha - C_{23}\psi$

$N = C_{32}\alpha - C_{33}\psi$

where X, Y, N, are defined as in 1.1 above.

C_{ij} = coefficients of linear creep

ξ = longitudinal creep ratio = longitudinal creep speed/forward speed

α = lateral creep ratio = (lateral speed /forward speed) - steer angle

ψ = spin creep ratio = spin angular velocity/forward speed

2. Plane Motion in a Moving Coordinate Frame

$\ddot{\mathbf{R}}_{O_1} = (\dot{u} - v\Omega)\mathbf{i} + (\dot{v} + u\Omega)\mathbf{j}$

$(\mathbf{i}, \mathbf{j}, \mathbf{k})$ axis system fixed to body at point O_1

where

u = speed of point O_1 in $\mathbf{i}$ direction

v = speed of point O_1 in $\mathbf{j}$ direction

$\Omega\mathbf{k}$ = absolute angular velocity of body

3. Routh-Hurwitz stability criteria

$$\left(a_2\frac{d^2}{dt^2} + a_1\frac{d}{dt} + a_0\right)y = x(t)$$

Stable if all $a_i > 0$

$$\left(a_3\frac{d^3}{dt^3} + a_2\frac{d^2}{dt^2} + a_1\frac{d}{dt} + a_0\right)y = x(t)$$

Stable if (i) all $a_i > 0$
and also (ii) $a_1 a_2 > a_0 a_3$

$$\left(a_4\frac{d^4}{dt^4} + a_3\frac{d^3}{dt^3} + a_2\frac{d^2}{dt^2} + a_1\frac{d}{dt} + a_0\right)y = x(t)$$

Stable if (i) all $a_i > 0$
and also (ii) $a_1 a_2 a_3 > a_0 a_3^2 + a_4 a_1^2$

DATA ON POTENTIAL THEORY AND ORBITS

1 For a distribution of mass with density $\rho(\mathbf{r})$ the gravitational potential U satisfies Poisson's equation

$$\nabla^2 U = -4\pi G\rho$$

where G is the gravitational constant (= 6.67×10^{-11} $\mathrm{Nm^2kg^{-2}}$). The gravitational force $\mathbf{F}$ experienced by a unit mass is given by

$$\mathbf{F} = \nabla U .$$

2 *In vacuo* $\rho = 0$, so that U satisfies Laplace's equation

$$\nabla^2 U = 0 .$$

3 For a point mass M at the origin

$$U(\mathbf{r}) = GM/|\mathbf{r}| .$$

For a general distribution of matter

$$U(\mathbf{r}) = G\iiint\frac{\rho(\mathbf{x})\mathrm{d}^3\mathbf{x}}{|\mathbf{r}-\mathbf{x}|} .$$

For a thin spherical shell of radius a and mass dM

$$U(r) = \begin{cases} GdM/|r|, & r > a \\ GdM/a, & r < a \end{cases}$$

4 Equations of motion for a particle in a plane orbit, in plane polar coordinates (r,θ):

$$\ddot{r} - r\dot{\theta}^2 = f_r \qquad r\ddot{\theta} + 2\dot{r}\dot{\theta} = f_\theta$$

where f_r, f_θ are the radial and transverse force components, per unit mass.

If $f_\theta = 0$ (i.e. for a central force) the second equation leads to conservation of angular momentum:

$$r^2\dot{\theta} = h = constant .$$

5 For a central force, the substitution $u = 1/r$ leads to an equation for the *shape* of the orbit, expressed as $u = u(\theta)$. The central force (assumed attractive) is described by a function $f(u)$ per unit mass, and for a given angular momentum per unit mass h the orbit satisfies

$$\frac{\partial^2 u}{\partial\theta^2} + u = \frac{f(u)}{h^2u^2} .$$

6 The equation of an ellipse in polar coordinates (r,θ) relative to a focus is

$$r = \frac{L}{(1+e\cos\theta)}$$ where e is the eccentricity.

The semi-major axis is $a = L/(1-e^2)$, the semi-minor axis is $b = L/\sqrt{1-e^2}$.

7 The mean anomaly M, the eccentric anomaly E and the true anomaly θ are related by

$$M = E - e\sin E \quad \text{and} \quad \cos\theta = \frac{\cos E - e}{1 - e\cos E}$$ where e is the eccentricity.

8 Spherical polar coordinates. Define (r,θ,ϕ) so that r is radial distance, θ is angle from the polar axis (co-latitude) and ϕ is the angle of longitude. Then:

$$\nabla U = \frac{\partial U}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\mathbf{e}_\phi$$

$$\nabla^2 U = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial U}{\partial r}\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial U}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2}$$

9 Axisymmetric solutions to Laplace's equation arising from separation of variables in spherical polar coordinates are

$$U(r,\theta) = \begin{cases} r^n P_n(\cos\theta) \\ r^{-n-1}P_n(\cos\theta) \end{cases}$$

where P_n is the Legendre polynomial of order n, describing the nth zonal harmonic. The first few Legendre polynomials are as follows:

$$P_0(\xi) = 1 \quad P_1(\xi) = \xi \quad P_2(\xi) = (3\xi^2 - 1)/2$$

$$P_3(\xi) = (5\xi^3 - 3\xi)/2 \quad P_4(\xi) = (35\xi^4 - 30\xi^2 + 3)/8 \cdot$$

10 The external potential of the Earth can be expressed as a sum of spherical-harmonic contributions. We consider in detail only the effect of the zonal harmonics, whose contribution can be written in standard form

$$U(r,\theta) = \frac{\mu}{r}\left[1 - \sum_{n=2}^{\infty}(R/r)^n J_n P_n(\cos\theta)\right].$$

For the Earth, $\mu = G\times M_{Earth} = 398603\ km^3s^{-2}$, mean radius $R = 6378$ km,

$$J_2 = 1082\times10^{-6}, \quad J_3 = -2.55\times10^{-6}, \quad J_4 = -1.65\times10^{-6}$$

Gravitational mass of the sun = 332946μ, gravitational mass of the moon = μ/81.3

Mean radius of Earth's orbit = 1.496×10^8 km, that of moon's orbit = 384400 km.