ENGINEERING TRIPOS PART IIB

Wednesday 22 April 2009 2.30 to 4

Module 4C9

CONTINUUM MECHANICS

Answer not more than **two** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
4C9 datasheet (6 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) If $e_{\alpha\beta\gamma}$ is the alternating tensor:

(i) demonstrate that $e_{inm}e_{inm} = 6$; [15%]

(ii) simplify the expressions $e_{1jk}\delta_{3j}v_k$, $e_{inm}e_{inq}$ and $e_{ijk}e_{lmk}v_{m,jl}$. [35%]

(b) Figure 1, which is drawn to scale, shows part of a slip line field. At the origin, the state of stress is $\sigma_{xx} = 15$ MPa, $\tau_{xy} = 50$ MPa and $\sigma_{yy} = 185$ MPa . Use the Hencky equations to estimate the stress at point A. [50%]

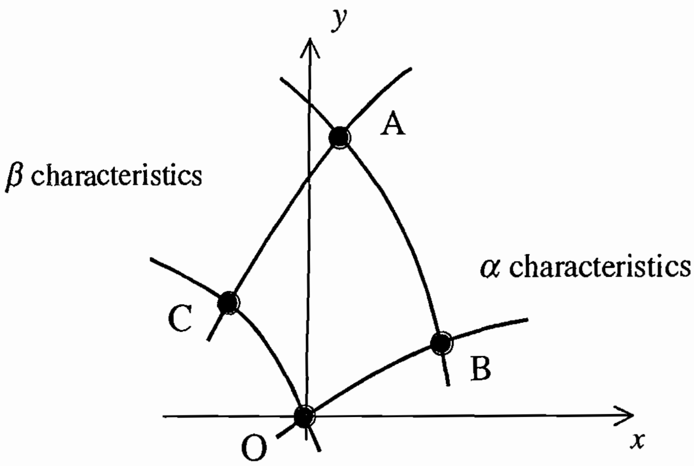

Fig. 1

2 A thin, square-section, metal wire is stretched in a tensile testing machine. Between the grips of the machine, the wire passes through a sealed pressure vessel which can apply any constant compressive pressure p. The testing machine loads the wire by applying a stress σ_{zz} which increases linearly with time t so that $\sigma_{zz} = \alpha t$.

(a) Write out the full deviatoric stress tensor for that length of the wire which is inside the pressure vessel and show that the Von Mises equivalent stress is given by the expression $\bar{\sigma} = p + \sigma_{zz}$. [35%]

(b) The wire obeys the Swift hardening law $Y = C(\varepsilon_0 + \bar{\varepsilon})^n$ where Y is the yield stress and ε_0, C and n are material constants. It does not yield until $\bar{\sigma} > C(\varepsilon_0)^n$. Derive a formula for the strain ε_{zz} of the wire along its length as a function of time. [35%]

(c) Sketch a graph illustrating the way in which ε_{zz} grows with time showing clearly the effect of increasing the pressure p from zero. [30%]

3 Figure 2(a) shows a curved beam of constant cross-section whose outer surface is an arc of radius b and inner surface an arc of radius a. The angle subtended by these two arcs is $\pi/2$. The two ends of the beam are subject only to bending moments of magnitude M as indicated.

(a) Explain why there will be a bending moment of magnitude M but zero axial and shear forces for all values of θ between 0 and $\pi/2$ and why this implies that a suitable Airy stress function is

$$\phi = Ar^2 + Br^2 \ln r + C \ln r + D\theta$$

[20%]

(b) Verify that using this function

$$\sigma_{\theta\theta} = 2A + B(2\ln r + 3) - \frac{C}{r^2}$$

and obtain the corresponding expressions for σ_{rr} and $\sigma_{r\theta}$. [20%]

(c) Explain the steps whereby the constants A, B, C and D may be evaluated from the expressions found in part (b). If

$$A = \frac{M}{N}\left(b^2 - a^2 + 2b^2 \ln b - 2a^2 \ln a\right)$$

$$B = -\frac{2M}{N}\left(b^2 - a^2\right)$$

$$C = -\frac{4M}{N} a^2 b^2 \ln(b/a)$$

find an expression for N in terms of the dimensions a and b. [35%]

(d) Explain how this analysis might be used to estimate the maximum tensile stress on the surface of the fillet, which has radius a, and which forms part of the component shown in Fig. 2(b). The component has uniform thickness t and the dimension $b = na$. Obtain an expression for this estimate of stress in terms of the applied moment M, the dimensions a and t, and the ratio n. [25%]

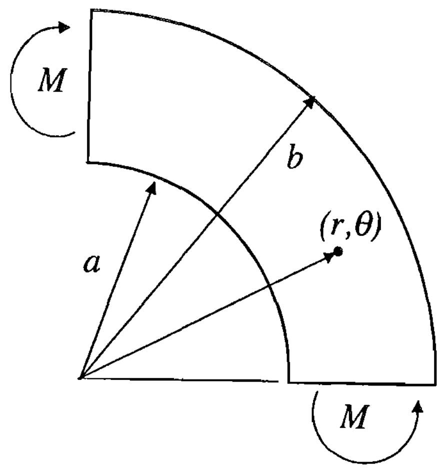

Fig. 2(a)

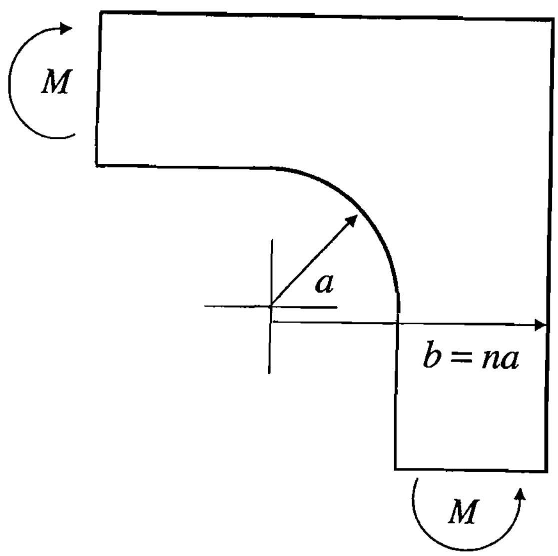

Fig. 2(b)

END OF PAPER

ENGINEERING TRIPOS Part IIB

Module 4C9 Data Sheet

SUBSCRIPT NOTATION

Repeated suffix implies summation

$\underset{\sim}{a} = a_1 \underset{\sim}{e}_1 + a_2 \underset{\sim}{e}_2 + a_3 \underset{\sim}{e}_3$	$a_i e_i$
$\underset{\sim}{a} \bullet \underset{\sim}{b}$	$a_i b_i \equiv a_i b_j \delta_{ij}$
$\underset{\sim}{c} = \underset{\sim}{a} \times \underset{\sim}{b}$	$c_i = e_{ijk} a_j b_k$
$\underset{\sim}{d} = \underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c})$	$d_k = -e_{ijk} e_{irs} a_j b_r c_s = a_j b_k c_j - a_i b_i c_k$
Kronecker delta δ_{ij}	$\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$
e_{ijk}	$e_{ijk} = 1$ when indices cyclic; $= -1$ when indices anticyclic and $= 0$ when any indices repeat
$e - \delta$ identity	$e_{ijk} e_{ilm} \equiv \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$
trace$a$	tr$a = a_{ii} = a_{11} + a_{22} + a_{33}$
$\dfrac{\partial \sigma_{ij}}{\partial x_i} = \dfrac{\partial \sigma_{1j}}{\partial x_1} + \dfrac{\partial \sigma_{2j}}{\partial x_2} + \dfrac{\partial \sigma_{3j}}{\partial x_3}$	$\sigma_{ij,i}$
grad$\phi = \nabla \phi$	$\dfrac{\partial \phi}{\partial x_i} = \phi_{,i}$
div$\underset{\sim}{V}$	$V_{i,i}$
curl$\underset{\sim}{V} \equiv \underset{\sim}{\nabla} \times \underset{\sim}{V}$	$e_{ijk} V_{k,j}$

<u>Rotation of Orthogonal Axes</u>

If $01'2'3'$ is related to 0123 by rotation matrix a_{ij}

vector v_i becomes	$v'_\alpha = a_{\alpha i} v_i$
tensor σ_{ij} becomes	$\sigma'_{\alpha\beta} = a_{\alpha i} a_{\beta j} \sigma_{ij}$

Evaluation of principal stresses

deviatoric stress $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$

$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$

$$I_1 = \sigma_{ii} = \mathrm{tr}\sigma$$
$$I_2 = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right)$$
$$I_3 = \frac{1}{6}\left(e_{ijk}e_{pqr}\sigma_{ip}\sigma_{jq}\sigma_{kr}\right)$$

$s^3 - J_1 s^2 + J_2 s - J_3 = 0$

$$J_1 = s_{ii} = \mathrm{tr}s \;;\; J_2 = \frac{1}{2}s_{ij}s_{ij} \;\;;\; J_3 = \frac{1}{3}s_{ij}s_{jk}s_{ki}$$

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equilibrium $\quad \sigma_{ij},_i + b_j = 0$

small strains $\quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

compatibility $\quad \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{lj,ki} - \varepsilon_{ki,lj} e_{pik}e_{qjl}\varepsilon_{ij,kl} = 0$

equivalent to $\quad e_{pik}e_{qjl}\varepsilon_{ij,kl} \equiv e_{pik}e_{qjl}\frac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} = 0$

Linear elasticity $\quad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

Hooke's law $\quad E\varepsilon_{ij} = (1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}$

Lamé's equations $\quad \sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

equivalent stress $\quad \bar{\sigma} = \sqrt{\frac{3}{2}s_{ij}s_{ij}} = \sqrt{3J_2}$

equivalent strain increment $\quad \mathrm{d}\bar{\varepsilon} = \sqrt{\frac{2}{3}\mathrm{d}\varepsilon_{ij}\mathrm{d}\varepsilon_{ij}}$

Elastic torsion of prismatic bars

Warping function $\Psi(x_1, x_2)$ satisfies $\nabla^2\Psi = \Psi_{,ii} = 0$

If Prandtl stress function $\phi(x_1, x_2)$ satisfies $\nabla^2\phi = \phi_{,ii} = -2G\alpha$ where α is the twist per unit length then

$$\sigma_{31} = \phi,_2 = \frac{\partial\phi}{\partial x_2} \;,\; \sigma_{32} = -\phi,_1 = -\frac{\partial\phi}{\partial x_1} \;\text{ and }\; T = 2\iint_A \phi(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2$$

Equivalence of elastic constants

	E	ν	$G=\mu$	λ
E, ν	–	–	$\dfrac{E}{2(1+\nu)}$	$\dfrac{\nu E}{(1+\nu)(1-2\nu)}$
E, G	–	$\dfrac{E-2G}{2G}$	–	$\dfrac{(2G-E)G}{E-3G}$
E, λ	–	$\dfrac{E-\lambda+R}{4\lambda}$	$\dfrac{E-3\lambda+R}{4}$	–
ν, G	$2G(1+\nu)$	–	–	$\dfrac{2G\nu}{1-2\nu}$
ν, λ	$\dfrac{\lambda(1+\nu)(1-2\nu)}{\nu}$	–	$\dfrac{\lambda(1-2\nu)}{2\nu}$	–
G, λ	$\dfrac{G(3\lambda+2G)}{\lambda+G}$	$\dfrac{\lambda}{2(\lambda+G)}$	–	–

$R=\sqrt{E^2+2E\lambda+9\lambda^2}$

Two-dimensional Airy Stress function

Biharmonic equation $\nabla^4\phi \equiv \phi_{,\alpha\alpha\beta\beta}=0$

Stresses $\sigma_{\alpha\beta}=e_{\alpha\gamma}\, e_{\beta\delta}\, \phi_{,\gamma\delta}$

$$\text{where } e_{\alpha\beta} \equiv e_{3\alpha\beta} = \begin{cases} 1 \text{ if } \alpha=1, \beta=2 \\ 0 \text{ if } \alpha=\beta \\ -1 \text{ if } \alpha=2, \beta=1 \end{cases}$$

Plane stress and plane strain

$$G\varepsilon_{11}=\frac{1}{8}\{\sigma_{11}(1+\kappa)+\sigma_{22}(\kappa-3)\}$$

$$G\varepsilon_{22}=\frac{1}{8}\{\sigma_{22}(1+\kappa)+\sigma_{11}(\kappa-3)\}$$

$$G\varepsilon_{12}=\frac{\sigma_{12}}{2}$$

$$\text{where } \begin{cases} \kappa=(3-\nu)/(1+\nu) & \text{in plane stress and} \\ \kappa=3-4\nu & \text{in plane strain} \end{cases}$$

Plasticity

von Mises yield criterion

$$f=\frac{3}{2}\sigma'_{ij}\sigma'_{ij}-Y^2\equiv\frac{3}{2}s_{ij}s_{ij}-Y^2$$

generalized flow rule

$$\dot{\varepsilon}_{ij}=\dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}}$$

Slip Line Fields

Henky equations

$$p-2k\phi=\text{constant along } \alpha \text{ line}$$

$$p+2k\phi=\text{constant along } \beta \text{ line}$$

Geiringer equations

$$\frac{\mathrm{d}v_\alpha}{\mathrm{d}s}=v_\beta\frac{\mathrm{d}\phi}{\mathrm{d}s} \quad \text{along } \alpha \text{ line}$$

$$\frac{\mathrm{d}v_\beta}{\mathrm{d}s}=v_\alpha\frac{\mathrm{d}\phi}{\mathrm{d}s} \quad \text{along } \beta \text{ line}$$

Table I – The Michell solutions — stress components

$\phi(r,\theta)$	σ_{rr}	$\sigma_{\theta\theta}$	$\sigma_{r\theta}$
r^2	2	2	0
$r^2 \ln r$	$2\ln r + 1$	$2\ln r + 3$	0
$\ln r$	$1/r^2$	$-1/r^2$	0
θ	0	0	$1/r^2$
$r^3\cos\theta$	$2r\cos\theta$	$6r\cos\theta$	$2r\sin\theta$
$r\theta\sin\theta$	$2\cos\theta/r$	0	0
$r\ln r\cos\theta$	$\cos\theta/r$	$\cos\theta/r$	$\sin\theta/r$
$\cos\theta/r$	$-2\cos\theta/r^3$	$2\cos\theta/r^3$	$-2\sin\theta/r^3$
$r^3\sin\theta$	$2r\sin\theta$	$6r\sin\theta$	$-2r\cos\theta$
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\ln r\sin\theta$	$\sin\theta/r$	$\sin\theta/r$	$-\cos\theta/r$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\sin\theta/r^3$	$2\cos\theta/r^3$
$r^{n+2}\cos n\theta$	$-(n+1)(n-2)r^n\cos n\theta$	$(n+1)(n+2)r^n\cos n\theta$	$n(n+1)r^n\sin n\theta$
$r^{-n+2}\cos n\theta$	$-(n+2)(n-1)r^{-n}\cos n\theta$	$(n-1)(n-2)r^{-n}\cos n\theta$	$-n(n-1)r^{-n}\sin n\theta$
$r^n\cos n\theta$	$-n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\cos n\theta$	$n(n-1)r^{n-2}\sin n\theta$
$r^{-n}\cos n\theta$	$-n(n+1)r^{-n-2}\cos n\theta$	$n(n+1)r^{-n-2}\cos n\theta$	$-n(n+1)r^{-n-2}\sin n\theta$
$r^{n+2}\sin n\theta$	$-(n+1)(n-2)r^n\sin n\theta$	$(n+1)(n+2)r^n\sin n\theta$	$-n(n+1)r^n\cos n\theta$
$r^{-n+2}\sin n\theta$	$-(n+2)(n-1)r^{-n}\sin n\theta$	$(n-1)(n-2)r^{-n}\sin n\theta$	$n(n-1)r^{-n}\cos n\theta$
$r^n\sin n\theta$	$-n(n-1)r^{n-2}\sin n\theta$	$n(n-1)r^{n-2}\sin n\theta$	$-n(n-1)r^{n-2}\cos n\theta$
$r^{-n}\sin n\theta$	$-n(n+1)r^{-n-2}\sin n\theta$	$n(n+1)r^{-n-2}\sin n\theta$	$n(n+1)r^{-n-2}\cos n\theta$

Table II – The Michell solutions — displacement components

For plane strain $\kappa = 3 - 4\nu$; for planes stress $\kappa = (3-\nu)/(1+\nu)$

$\phi(r,\theta)$	$2Gu_r$	$2Gu_\theta$
r^2	$(\kappa - 1)r$	0
$r^2 \ln r$	$(\kappa - 1)r \ln r - r$	$(\kappa + 1)r\theta$
$\ln r$	$-1/r$	0
θ	0	$-1/r$
$r^3 \cos\theta$	$(\kappa - 2)r^2 \cos\theta$	$(\kappa + 2)r^2 \sin\theta$
$r\theta \sin\theta$	$0.5[(\kappa-1)\theta\sin\theta - \cos\theta + (\kappa+1)\ln r\cos\theta]$	$0.5[(\kappa-1)\theta\cos\theta - \sin\theta - (\kappa+1)\ln r\sin\theta]$
$r \ln r \cos\theta$	$0.5[(\kappa+1)\theta\sin\theta - \cos\theta + (\kappa-1)\ln r\cos\theta]$	$0.5[(\kappa+1)\theta\cos\theta - \sin\theta - (\kappa-1)\ln r\sin\theta]$
$\cos\theta / r$	$\cos\theta / r^2$	$\sin\theta / r^2$
$r^3 \sin\theta$	$(\kappa - 2)r^2 \sin\theta$	$-(\kappa - 2)r^2 \cos\theta$
$r\theta \cos\theta$	$0.5[(\kappa-1)\theta\cos\theta + \sin\theta - (\kappa+1)\ln r\sin\theta]$	$0.5[-(\kappa-1)\theta\sin\theta - \cos\theta - (\kappa+1)\ln r\cos\theta]$
$r \ln r \sin\theta$	$0.5[-(\kappa+1)\theta\cos\theta - \sin\theta + (\kappa-1)\ln r\sin\theta]$	$0.5[(\kappa+1)\theta\sin\theta + \cos\theta + (\kappa-1)\ln r\cos\theta]$
$\sin\theta / r$	$\sin\theta / r^2$	$-\cos\theta / r^2$
$r^{n+2} \cos n\theta$	$(\kappa - n - 1)r^{n+1} \cos n\theta$	$(\kappa + n + 1)r^{n+1} \sin n\theta$
$r^{-n+2} \cos n\theta$	$(\kappa + n - 1)r^{-n+1} \cos n\theta$	$-(\kappa - n + 1)r^{-n+1} \sin n\theta$
$r^n \cos n\theta$	$-nr^{n-1} \cos n\theta$	$nr^{n-1} \sin n\theta$
$r^{-n} \cos n\theta$	$nr^{-n-1} \cos n\theta$	$nr^{-n-1} \sin n\theta$
$r^{n+2} \sin n\theta$	$(\kappa - n - 1)r^{n+1} \sin n\theta$	$-(\kappa + n + 1)r^{n+1} \cos n\theta$
$r^{-n+2} \sin n\theta$	$(\kappa + n - 1)r^{-n+1} \sin n\theta$	$(\kappa - n + 1)r^{-n+1} \cos n\theta$
$r^n \sin n\theta$	$-nr^{n-1} \sin n\theta$	$-nr^{n-1} \cos n\theta$
$r^{-n} \sin n\theta$	$nr^{-n-1} \sin n\theta$	$-nr^{-n-1} \cos n\theta$

JAW/JMA

Answers

1 (a) (i) 6 (ii) $-v_k$; 6 ; $v_{j,ji} - v_{i,jj}$

(b) $\phi_0 \approx 30°, \phi_B \approx 15°, \phi_A \approx 45°$

$$\text{at A} \quad \begin{cases} \sigma_{xx} = -157 \text{ MPa} \\ \sigma_{yy} = 43 \text{ MPa} \\ \tau_{xy} = 0 \end{cases}$$

2 (a) $\sigma'_{ij} = \dfrac{p + \sigma_{zz}}{3} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(b) $\varepsilon_{zz} = \bar{\varepsilon} = \varepsilon_0 \left[\left(\dfrac{p + \alpha t}{p + \alpha t_1} \right)^{\frac{1}{n}} - 1 \right]$ for $t > t_1$ otherwise 0

3 $\sigma_{rr} = 2A + B(2\ln r + 1) + \dfrac{C}{r^2}; \quad \sigma_{r\theta} = \dfrac{D}{r^2}$

(c) $N = \left(b^2 - a^2\right)^2 - 4a^2b^2\left(\ln(b/a)\right)^2$

(d) $\sigma_{\theta\theta} = \dfrac{4\left(2n^2 \ln n - n^2 + 1\right)}{\left(n^2 - 1\right)^2 - 4n^2 (\ln n)^2} \times \dfrac{M}{a^2 t}$

$$\sigma_{\theta\theta} = \frac{6M}{(n-1)^2 a^2 t}$$