ENGINEERING TRIPOS PART IIB

Wednesday 22 April 2009 2.30 to 4

Module 4F1

CONTROL SYSTEM DESIGN

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: Formulae sheet (3 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed
Supplementary pages: Two extra copies of Fig. 1 (Question 3).

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) Let $G(s)$ be a rational transfer-function.

(i) Describe how the Nyquist and root-locus diagrams of $G(s)$ can be viewed in terms of a single mapping between two complex planes. [10%]

(ii) State the definition and properties of a conformal mapping. [15%]

(iii) Briefly describe an informal proof of the Nyquist stability criterion which makes use of conformal mappings. [15%]

(b) Let

$$G(s) = \frac{s+1}{(s-1)^2(s+7)}.$$

(i) Sketch the root-locus diagrams for $k > 0$ and $k < 0$ on separate diagrams. [15%]

(ii) Sketch the complete Nyquist diagram for $G(s)$. [15%]

(iii) Find the imaginary axis crossing points in the root-locus diagrams and the real axis crossing points of the Nyquist diagram. [15%]

(iv) Determine the number of closed-loop poles in the right half plane when $G(s)$ is connected in the standard negative feedback configuration with gain k, for all values of k both positive and negative. Confirm that the Nyquist and root-locus diagrams give the same assessment for closed-loop stability. [15%]

2 (a) State the small gain theorem. An uncertain system is modelled as:

$$G_1(s) = \frac{G(s)}{1+\Delta(s)G(s)}$$

where $G(s)$ is a known transfer function and $\Delta(s)$ is assumed only to be stable and satisfy a bound $|\Delta(j\omega)| \leq h(\omega)$ for all ω. Let $K(s)$ stabilise $G(s)$ in a unity gain negative feedback system. Derive a necessary and sufficient condition for $K(s)$ to stabilise $G_1(s)$. [25%]

(b) Consider a plant with transfer function

$$\frac{s+3}{s^2(s+3)+ae^{-sT}} \qquad (1)$$

where $T \geq 0$ is unknown and $|a| \leq a_0$ where a_0 is unknown. Suppose the controller $K(s) = \dfrac{3s+1}{s+3}$ is employed in the standard unity gain negative feedback configuration.

(i) Using the result of part (a) find conditions on a_0 and T which guarantee closed-loop stability. [30%]

(ii) Assuming $T = 0$ find directly the range of a in (1) for which the closed-loop system is stable. Hence comment on whether the value of a_0 derived in part (b)(i) is conservative. [25%]

(c) For the plant and controller of part (b) with $a = 0$, design a two-degree of freedom control system so that the closed-loop transfer function from reference input to plant output is $1/(s+1)^2$. [20%]

3 Fig. 1 is the Bode diagram of a system $G(s)$ for which a feedback compensator $K(s)$ in the standard negative feedback configuration is to be designed. It may be assumed that $G(s)$ is a real-rational transfer function, and that all poles and zeros have moduli which lie within the range of frequencies shown on the diagram.

(a) (i) Sketch on a copy of Fig. 1 the expected phase of $G(j\omega)$ if $G(s)$ were stable and minimum phase. [10%]

(ii) Determine whether $G(s)$ has any right half plane poles or zeros (it doesn't have both), and estimate their location (if there are any). [10%]

(b) Let $S(s)$ and $T(s)$ denote the sensitivity and complementary sensitivity functions. Find a feedback compensator $K(s)$ which provides internal stability of the closed-loop and satisfies the following specifications:

A: $|G(j\omega)K(j\omega)| = 1$ at $\omega = 2$ rad/sec,

B: a phase margin of at least $45°$.

C: $S(0) = 0$,

D: $|T(j\omega)| \leq 0.01$ for $\omega \geq 30$ rad/sec.

Show on another copy of Fig. 1 the effect of this compensator on the return-ratio transfer function. [50%]

(c) Suppose it is desired to increase the crossover frequency in Specification A for the design of part (b). Briefly discuss the likely limitations if:

(i) All other specifications are left the same. [15%]

(ii) Specification D is removed. [15%]

Two copies of Fig. 1 are provided on separate sheets. These should be handed in with your answers.

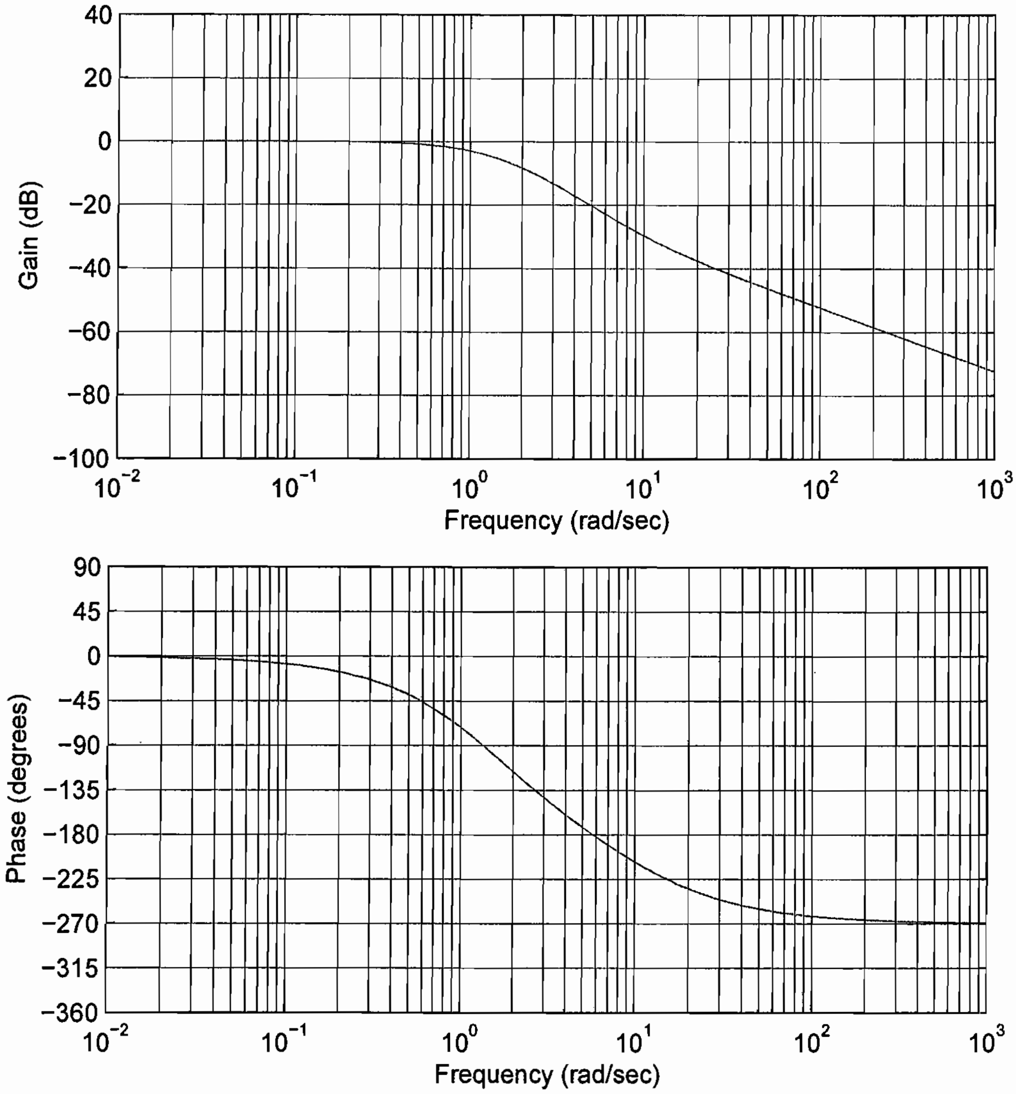
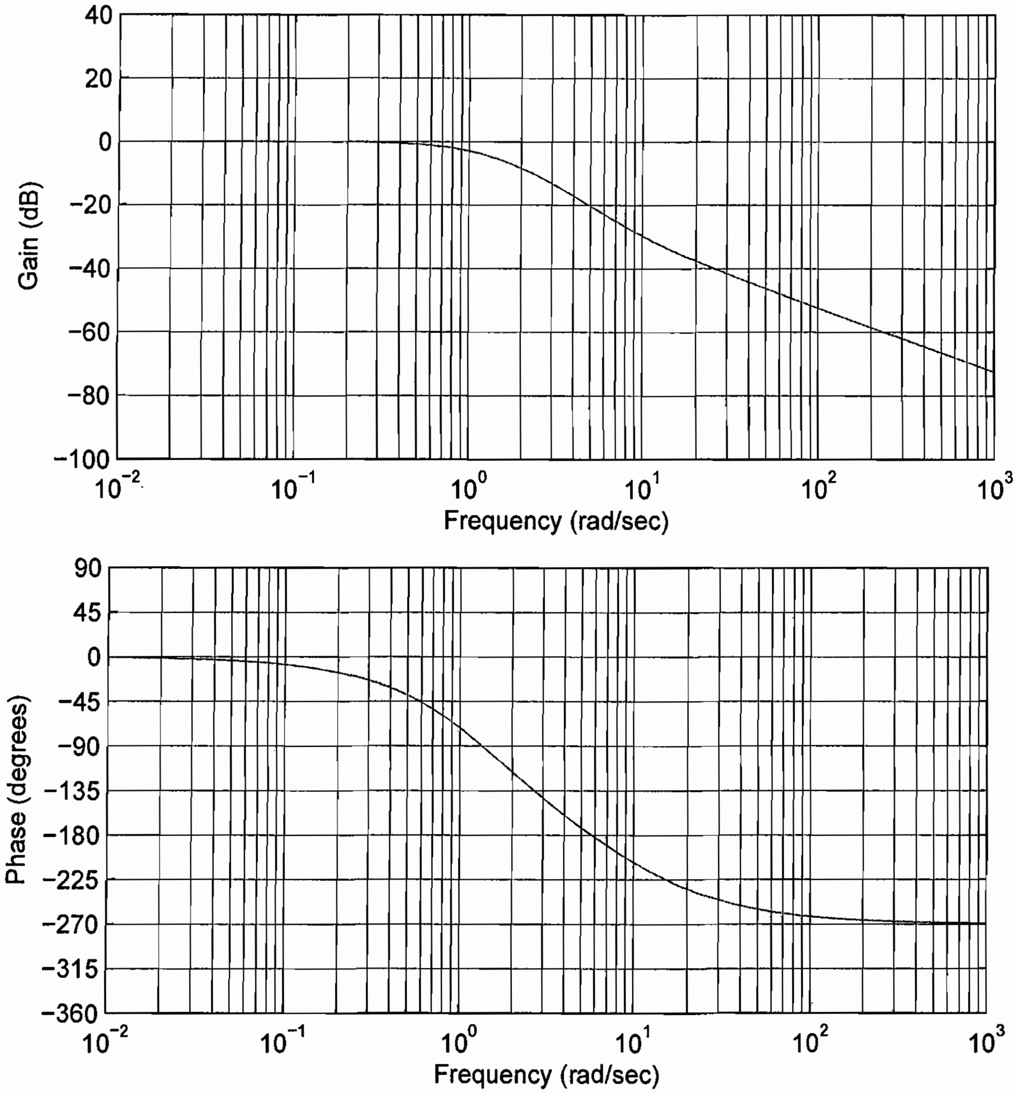


Fig. 1

END OF PAPER

ENGINEERING TRIPOS PART IIB

Wednesday 22 April 2009, Module 4F1, Question 3.



Extra copy of Fig. 1: Bode diagram of $G(s)$ for Question 3.

Formulae sheet for Module 4F1: Control System Design

To be available during the examination.

1 Terms

For the standard feedback system shown below, the **Return-Ratio Transfer Function** $L(s)$ is given by

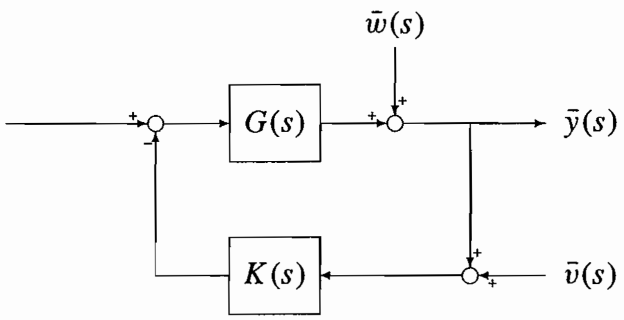
$$L(s) = G(s)K(s),$$

the **Sensitivity Function** $S(s)$ is given by

$$S(s) = \frac{1}{1 + G(s)K(s)}$$

and the **Complementary Sensitivity Function** $T(s)$ is given by

$$T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}$$



The closed-loop system is called **Internally Stable** if each of the *four* closed-loop transfer functions

$$\frac{1}{1 + G(s)K(s)}, \quad \frac{G(s)K(s)}{1 + G(s)K(s)}, \quad \frac{K(s)}{1 + G(s)K(s)}, \quad \frac{G(s)}{1 + G(s)K(s)}$$

are stable (which is equivalent to $S(s)$ being stable and there being no right half plane pole/zero cancellations between $G(s)$ and $K(s)$).
A transfer function is called **real-rational** if it can be written as the ratio of two polynomials in s, the coefficients of each of which are purely real.

2 Phase-lead compensators

The phase-lead compensator

$$K(s) = \alpha \frac{s + \omega_c/\alpha}{s + \omega_c \alpha}, \quad \alpha > 1$$

achieves its maximum phase advance at $\omega = \omega_c$, and satisfies:

$$|K(j\omega_c)| = 1, \quad \text{and} \quad \angle K(j\omega_c) = 2 \arctan \alpha - 90°.$$

3 The Bode Gain/Phase Relationship

If

1. $L(s)$ is a real-rational function of s,
2. $L(s)$ has no poles or zeros in the *open* RHP ($\mathrm{Re}(s) > 0$) and
3. satisfies the normalization condition $L(0) > 0$.

then

$$\boxed{\angle L(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d}{dv}\log|L(j\omega_0 e^v)|\log\coth\frac{|v|}{2}\,dv}$$

Note that

$$\log\coth\frac{|v|}{2} = \log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|, \text{ where } \omega = \omega_0 e^v.$$

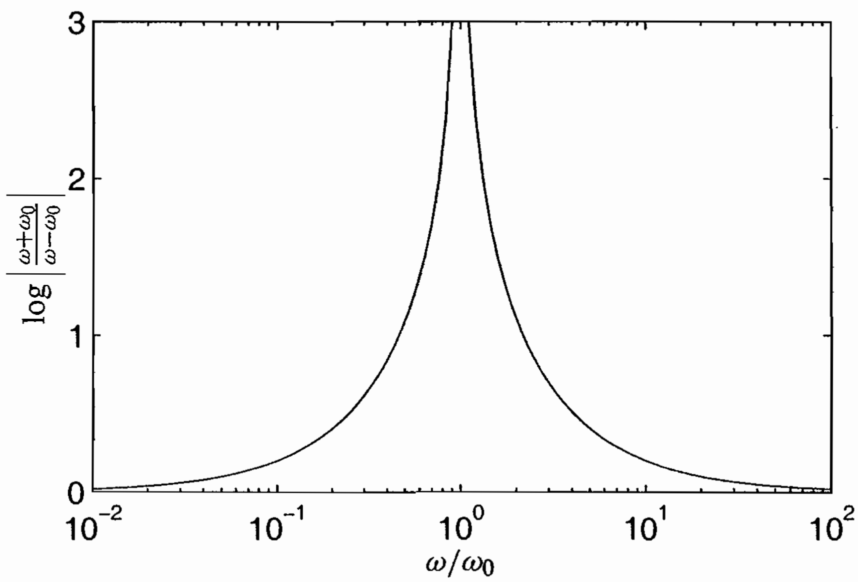


Figure 1:

If the slope of $L(j\omega)$ is approximately constant for a sufficiently wide range of frequencies around $\omega = \omega_0$ we get the *approximate form of the Bode Gain/Phase Relationship*

$$\angle L(j\omega_0) \approx \frac{\pi}{2}\left.\frac{d\log|L(j\omega_0 e^v|)}{dv}\right|_{\omega=\omega_0}.$$

4 The Poisson Integral

If $H(s)$ is a real-rational function of s which has no poles or zeros in $\mathrm{Re}(s) > 0$, then if $s_0 = \sigma_0 + j\omega_0$ with $\sigma_0 > 0$

$$\boxed{\log H(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} \log H(j\omega)\, d\omega}$$

and

$$\boxed{\log |H(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} \log \left|H\left(j|s_0|e^{v}\right)\right| dv}$$

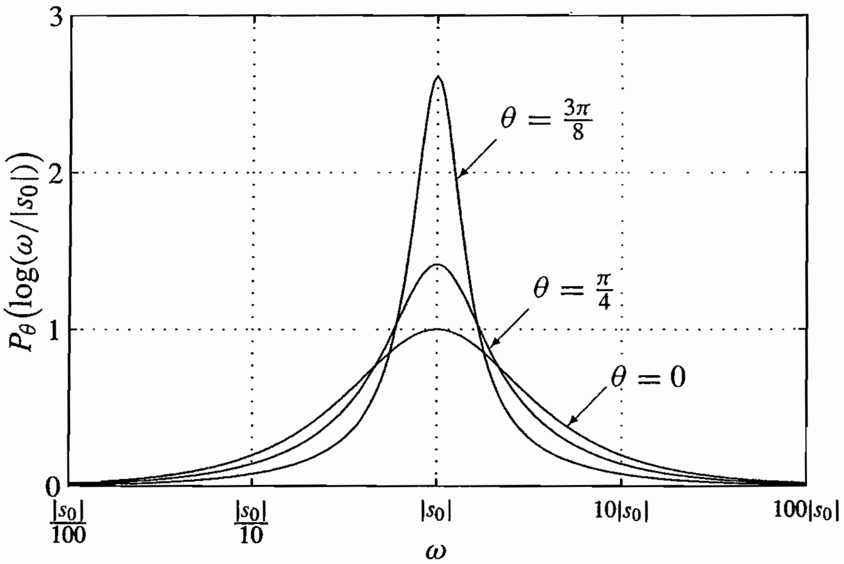
where $v = \log\left(\frac{\omega}{|s_0|}\right)$ and $\theta = \angle(s_0)$. Note that, if s_0 is real, so $\angle s_0 = 0$, then

$$\frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} = \frac{1}{\cosh v}.$$

We define

$$P_\theta(v) = \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta}$$

and give graphs of P_θ below.



The indefinite integral is given by

$$\int P_\theta(v)\, dv = \arctan\left(\frac{\sinh v}{\cos\theta}\right)$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} P_\theta(v)\, dv = 1 \quad \text{for all } \theta.$$

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M.C. Smith
November 2002

4F1 2009 — Answers

1(b)(iii) Imaginary axis crossings of root-locus: 0, $\pm\sqrt{5}$. Real axis crossings of Nyquist diagram: $1/7$, $-1/18$.

2(b)(i) $a_0 < 1$, no condition on T.

2(b)(ii) $-1 < a < 8$.

3(a)(ii) $G(s)$ has one right half plane zero.

M.C. Smith, 11 May 2009