ENGINEERING TRIPOS PART IIB

Wednesday 29 April 2009 2.30 to 4

Module 4F2

ROBUST MULTIVARIABLE CONTROL

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) Consider the linear system

$$\dot{x} = Ax + Bw$$
$$z = Cx$$

where A is a stable matrix. Let $G(s)$ denote the transfer function of the system from w to z.

(i) State the definition of $\|G(s)\|_2$. Discuss an interpretation of $\|G(s)\|_2$ in terms of the norms of w and z. [15%]

(ii) Provide an expression for computing $\|G(s)\|_2$ using the solution of an appropriate Lyapunov equation. State conditions under which the Lyapunov equation has a solution, and under which the solution is positive definite. [15%]

(b) Consider the state feedback $\mathscr{H}_2$ optimal control problem

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{12} u$$
$$y = x$$

with (A, B_2) controllable, (A, C_1) observable and $D_{12}^T \begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}$.

(i) For an arbitrary stabilising controller $u = -Kx$,

$$\|\mathscr{F}_l(P(s), K(s))\|_2^2 = 2\pi \,\text{Trace}\left(B_1^T L B_1\right)$$

where $L = L^T \geq 0$ is the solution to a particular Lyapunov equation and $\mathscr{F}_l$ denotes the lower linear fractional transformation. Show that this Lyapunov equation is

$$(A - B_2K)^T L + L(A - B_2K) + C_1^T C_1 + K^T K = 0$$

[20%]

(ii) Assume $X = X^T > 0$ is the solution to the Riccati equation

$$XA + A^T X - XB_2B_2^T X + C_1^T C_1 = 0$$

Using the Lyapunov equation in part (b) (i), show that

$$(A - B_2K)^T (L - X) + (L - X)(A - B_2K) + \left(K - B_2^T X\right)^T \left(K - B_2^T X\right) = 0$$

Deduce that $L - X \geq 0$ if K is an arbitrary stabilising controller. [25%]

(cont.

(iii) Using this last fact, show that $K = B_2^T X$ is the controller that minimises $\|\mathscr{F}_l(P,K)\|_2$ and give the value for the minimum.

[*Hint*: Note that, if $X = X^T \geq 0$, then $\text{Trace}(X) \geq 0$.] [25%]

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2 Consider the closed-loop transfer function from $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ in Figure 1.

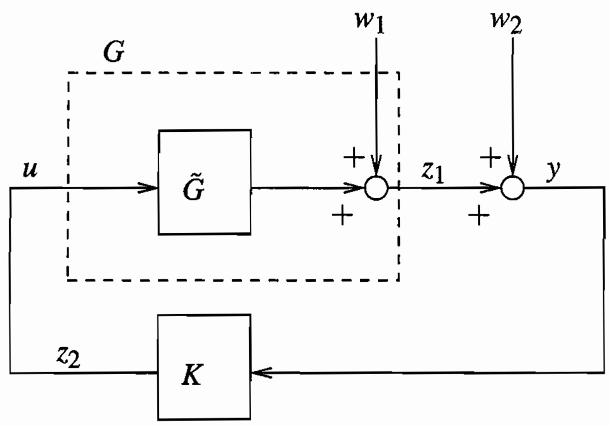

Fig. 1

(a) Find the generalised plant $P(s)$ such that $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathscr{F}_l\left(P(s), K(s)\right) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and evaluate $\mathscr{F}_l\left(P(s), K(s)\right)$, where $\mathscr{F}_l$ denotes the lower linear fractional transformation. [30%]

(b) If $G(s)$ has a state-space realisation

$$\begin{aligned} \dot{x} &= ax + ku + w_1, \quad x \in \mathbb{R} \\ z_1 &= x \end{aligned}$$

where $a > 0$ and $k > 0$, find a state-space realisation for $P(s)$, as defined in part (a), in the following form

$$\begin{aligned} \dot{x} &= Ax + B_1 w_1 + B_2 u \\ z &= \begin{bmatrix} C_1 x \\ u \end{bmatrix} \\ y &= C_2 x + w_2 \end{aligned}$$

[10%]

(cont.

(c) For $G(s)$ as in part (b) find

$$\min_{K(s) \text{ stabilising}} \|\mathscr{F}_l(P(s),K(s))\|_2$$

as a function of a and k. [25%]

[*Hint*: The CARE and FARE are

$$XA + A^T X + C_1^T C_1 - XB_2B_2^T X = 0$$

and

$$YA^T + AY + B_1B_1^T - YC_2^T C_2 Y = 0,$$

respectively, with appropriate A, B_1, B_2, C_1, C_2. Then,

$$\|\mathscr{F}_l(P(s),K(s))\|_2^2 = 2\pi\left(\text{trace}\left(B_1^T X B_1\right) + \text{trace}\left(FYF^T\right)\right),$$

where $F = B_2^T X$.]

(d) Give a state-space realisation for the optimal controller K in part (c), find its pole(s) and, hence, show that it is stable. [15%]

[*Hint*: The optimal controller K is given by

$$\begin{bmatrix} \dot{x}_k \\ u \end{bmatrix} = \begin{bmatrix} A - B_2F - HC_2 & -H \\ F & 0 \end{bmatrix} \begin{bmatrix} x_k \\ y \end{bmatrix}$$

where $H = YC_2^T$.]

(e) With the same optimal controller K, give a state-space realisation for the closed-loop system. Find the poles of the closed-loop system and confirm that it is stable.

[20%]

3 (a) Calculate the $\mathscr{H}_\infty$–norm of the transfer function

$$P(s) = \frac{s+1}{(s+2)^2}$$

[10%]

(b) State the small gain theorem for the robust stability of a system, $Y(s) = T(s)U(s)$, in feedback with an uncertain system, $U(s) = \Delta(s)Y(s)$, where $T(s)$ and $\Delta(s)$ are both stable, $T(s)$ is known, and $\Delta(s)$ satisfies $\|\Delta(s)\|_\infty < \varepsilon$. [10%]

(c) An uncertain system has a transfer function, $G(s) = (I - W(s)\Delta(s))^{-1}G_0(s)$, where $G_0(s)$ and $W(s)$ are known stable transfer functions and $\|\Delta(s)\|_\infty < 1$. Show that the feedback system, $Y(s) = G(s)U(s)$, $U(s) = -Y(s)$ will be stable for all $\Delta(s)$ if and only if the feedback system is stable with $\Delta(s) = 0$ and $\|(I + G_0(s))^{-1}W(s)\|_\infty \leq 1$. [30%]

(d) Let $G(s) = \frac{s+2}{(s+1)(s+2+\alpha)}$, where the unknown real parameter, α, satisfies $-A < \alpha < A$. Let the nominal transfer function be $G_0(s) = \frac{1}{s+1}$, and consider the feedback system $Y(s) = G(s)U(s)$, $U(s) = -Y(s)$.

(i) Determine the maximum value of A such that the feedback system is guaranteed to be stable for all $|\alpha| < A$. [10%]

(ii) Represent $G(s)$ as in part (c) and apply the result of part (c) to give a condition on A for stability of the feedback system. [20%]

(iii) Calculate $N(s)$ and $M(s)$ where $G_0(s) = N(s)/M(s)$ is a normalised coprime factorisation. Show that $G(s)$ can be expressed as $G(s) = N(s)/(M(s) + \Delta_M(s))$ and briefly discuss the relation between this representation of the uncertain system and perturbations measured in the gap metric. [20%]

END OF PAPER