ENGINEERING TRIPOS PART IIB

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Monday 4 May 2009 2.30 to 4

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Module 4F3

NONLINEAR AND PREDICTIVE CONTROL

*Answer not more than **three** questions.*

*All questions carry the same number of marks.*

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*There are no attachments.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

1 (a) Define the following for an autonomous system $\dot{x} = f(x)$:

(i) Equilibrium point. [5%]

(ii) Stable equilibrium point. [10%]

(iii) Asymptotically stable equilibrium point. [10%]

(iv) Domain of attraction of an asymptotically stable equilibrium point. [5%]

(b) Consider the system

$$\dot{x}_1 = -x_1 + h(x_3)$$
$$\dot{x}_2 = -h(x_3)$$
$$\dot{x}_3 = -f(x_1) + g(x_2) - h(x_3)$$

where $f(0) = g(0) = h(0) = 0$, $yf(y) > 0$, $yg(y) > 0$, $yh(y) > 0$ for $0 < |y| < \alpha$ and $f, g, h$ are Lipschitz continuous functions.

(i) By considering the function

$$V(x_1, x_2, x_3) = \int_0^{x_1} f(y)dy + \int_0^{x_2} g(y)dy + \int_0^{x_3} h(y)dy$$

show that the origin is an asymptotically stable equilibrium of the system. [40%]

(ii) For the case $\frac{df(y)}{dy} = \frac{dg(y)}{dy} = 0$ for $y = 0$ discuss whether a linearization of the system would be sufficient to deduce asymptotic stability of the origin.

[20%]

(iii) Is the origin also globally asymptotically stable? [10%]

2 (a) Show that the describing function of the saturation nonlinearity

$$f(e) = \begin{cases} 1 & \text{if } e > \delta \\ \frac{e}{\delta} & \text{if } |e| \leq \delta \\ -1 & \text{if } e < -\delta \end{cases}$$

with $\delta > 0$, is given by [30%]

$$N_1(E) = \begin{cases} \frac{1}{\delta}, & \text{if } E \leq \delta \\ \frac{2}{\pi\delta}\left[\sin^{-1}\left(\frac{\delta}{E}\right) + \frac{\delta}{E}\sqrt{1-\left(\frac{\delta}{E}\right)^2}\right] & \text{if } E > \delta \end{cases}$$

(b) Using your answer to part (a) with $\delta = 1$, find the describing function $N_2(E)$ for the dead-zone nonlinearity [20%]

$$g(e) = \begin{cases} e-1 & \text{if } e > 1 \\ 0 & \text{if } |e| \leq 1 \\ e+1 & \text{if } e < -1 \end{cases}$$

(c) Show that $0 \leq N_1(E) \leq \frac{1}{\delta}$. [20%]

(d) The nonlinearity $f(e)$ is connected in negative feedback with a linear system whose transfer function is

$$G(s) = \frac{k}{(s+1)^2}$$

(i) Is a limit cycle predicted by the describing function method when $k > 0$? [15%]

(ii) Find the values of $k$ for which the circle criterion would guarantee global asymptotic stability of the interconnection. [15%]

3 (a) In the standard formulation of model predictive control, constraints on the state vector are expressed as linear inequalities of the form

$$MX \leq m \qquad (1)$$

where $M$ is a constant matrix, $m$ is a constant vector, $X$ is the vector

$$X = [x_1^T, x_2^T, \ldots, x_N^T]^T$$

and $x_s$ is the predicted value of the state vector $s$ steps into the prediction horizon. Explain why it is important that constraints are expressed in this linear form. [30%]

(b) Constraints often arise in the form $|x^i| \leq \ell_i$, where $x^i$ denotes the $i$'th component of the vector $x$. Explain how such constraints may be put into the form (1). [20%]

(c) Two spacecraft are in orbit around a planet. The 'chaser' craft is using model predictive control to approach and dock with the 'target' craft, by firing thrusters which exert forces $F_y$ and $F_z$, as shown in Fig.1. When the two craft are in the same orbital plane and are close together, the target craft may be considered to be travelling in a straight line; the tangential and radial distances from the chaser craft to the target craft, $y$ and $z$ respectively, can then be defined as shown in Fig.1.

In order to dock successfully, the following constraints must be satisfied during the final approach:

$$|\dot{z}| \leq 0.01 \text{ ms}^{-1}$$
$$|z| \leq 0.1 \text{ m}$$

Write these constraints in the form (1), assuming that the prediction horizon is $N = 2$, and that the state vector is defined as [30%]

$$x = [y, \dot{y}, z, \dot{z}]^T$$

(cont.

(d) The dynamics of the chaser craft shown in Fig. 1 are given approximately by

$$x_{k+1} = Ax_k + Bu_k$$

where the input vector consists of the two thruster forces:

$$u_k = \begin{bmatrix} F_y \\ F_z \end{bmatrix}_k$$

Explain how the constraints (1) can be expressed as linear inequalities involving the vector of predicted inputs, and the latest measured state, when $N = 2$. [20%]

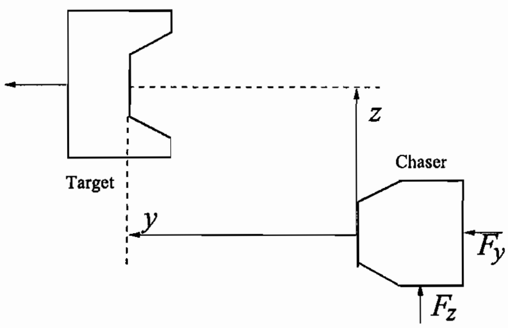


Fig. 1

4 (a) Explain the principle of operation of predictive control, and summarise its principal benefits and disadvantages. [30%]

(b) Predictive control is to be applied to the linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k)$$

with a prediction horizon of only one step and a cost function

$$V(x_0, u_0) = x_0^T Q x_0 + u_0^T R u_0 + x_1^T P x_1$$

where $x_0$ is the current measured state, $x_1$ is the predicted next state when input $u_0$ is applied, $P$, $Q$ and $R$ are all positive-definite, and

$$P = A^T PA + Q \qquad (2)$$

Let $u_0^*(x_0)$ denote the value of $u_0$ that minimises the value of $V(x_0, u_0)$, and let $V^*(x_0) = V(x_0, u_0^*)$ be the corresponding minimum value of $V$.

(i) Show that $V(Ax_0 + Bu_0^*, 0) < V^*(x_0)$, and hence that $V^*(Ax_0 + Bu_0^*) < V^*(x_0)$, if $x_0 \neq 0$. [40%]

(ii) Explain how this result can be used to prove stability of the closed loop when this predictive control is applied. [20%]

(iii) Explain why it can be deduced, from equation (2), that the open-loop system is stable. [10%]

**END OF PAPER**