ENGINEERING TRIPOS PART IIB

Wednesday 6 May 2009 9 to 10.30

Module 4F6

SIGNAL DETECTION AND ESTIMATION

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 a) Define *Information* and *Entropy* of a probability distribution and describe, in detail, how *Maximum Entropy* methods may be used to assign probability distributions. [30%]

b) Given the first moment of a distribution (from experimental measurements, for example) show, using Lagrange multipliers, that the distribution having *Maximum Entropy* is an Exponential distribution. [40%]

c) Derive an expression for the entropy of the Exponential distribution in terms of its standard deviation. [30%]

2 a) Define *Fisher Information* and *Cramer-Rao Lower Bound* and show that for an unbiased estimator, $\hat{\theta}(x)$, of a parameter θ, the variance associated with $\hat{\theta}(x)$ satisfies

$$var(\hat{\theta}(x)) \geq I_\theta^{-1}$$

where I_θ is the *Fisher Information* for the scalar parameter θ. [40%]

b) Derive the following condition for an efficient unbiased estimator:

$$\frac{\partial \ln p(x|\theta)}{\partial \theta} = I_\theta(\hat{\theta}(x) - \theta) \qquad (1)$$

where $p(x|\theta)$ is the likelihood function and $\hat{\theta}(x)$ is an estimator for θ.

[30%]

c) Describe how equation (1) leads to the *Neyman-Fisher* factorization theorem. [10%]

d) Using Bayesian reasoning, derive the *Neyman-Fisher* factorization theorem. [20%]

3 a) Define the terms *Error of the first kind* and *Error of the second kind* explaining the role they play in Detection Theory and explain the role played by the *log-likelihood ratio.* [40%]

b) In an M-ary digital transmission system the source signal can take one of M possible levels during a symbol period. In each symbol period the detector makes N measurements $\mathbf{y} = [y_1 y_2 \ldots y_N]^T$ of the channel output.

i) Show that the Maximum A-Posteriori (MAP) decision rule for the detector may be expressed as: [30%]

$$\text{Choose } H_i \text{ if } \max_{H_j}\{p(\mathbf{y}|H_j)P(H_j)\} = p(\mathbf{y}|H_i)P(H_i)\,.$$

ii) Show that the average error probability P_e for the detector is given by: [30%]

$$P_e = 1 - \sum_{i=1}^{M} P(D_i|H_i)P(H_i)$$

where:

$p(\mathbf{y}|H_i)$ is the probability density of the observation vector $\mathbf{y}$ conditional on hypothesis H_i being in force;

$P(H_i)$ is the *a-priori* probability of hypothesis H_i;

$P(D_i|H_i)$ is the probability of deciding in favour of hypothesis H_i when H_i is in force.

4 a) Describe, in detail, the Neyman-Pearson decision rule applied to detection theory and discuss the advantages and disadvantages of this decision rule over the MAP and Bayes criteria. [30%]

b) Define the *Receiver Operating Characteristic* of a detector and describe its role in both detection theory and data classification. [30%]

c) Show that the value of the threshold for the Neyman-Pearson test for a single observation is given by the slope of the receiver operating characteristic (ROC) at the required false alarm probability. [40%]

END OF PAPER