ENGINEERING TRIPOS PART IIA
ENGINEERING TRIPOS PART IIB

Monday 27th April 2009 2.30 to 4

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: Data Sheet for 4M12 (3 sides).

STATIONERY
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) Use index notation to show under what conditions $\varepsilon_{ijk} s_{jk} = 0$, where s_{jk} is an arbitrary second rank tensor. [10%]

(b) If $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are vectors, and

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \beta \boldsymbol{b} - \alpha \boldsymbol{a}$$

find expressions for α and β. [30%]

(c) In the advection-diffusion equation, the advective term is sometimes written as $\nabla . (\boldsymbol{a} \phi)$ and at other times as $\boldsymbol{a} . \nabla \phi$, where ϕ is a scalar and $\boldsymbol{a}$ is a vector.

(i) Show using index notation that $\nabla . (\boldsymbol{a} \phi) = \boldsymbol{a} . \nabla \phi$ if $\nabla . \boldsymbol{a} = 0$. [10%]

(ii) If $\nabla . \boldsymbol{a} = 0$, for a volume V, what are the other conditions on $\boldsymbol{a}$ under which

$$\int_V w \boldsymbol{a} \cdot \nabla \phi \, dV = -\int_V \nabla w \cdot \boldsymbol{a} \phi \, dV$$

will hold for all functions w. [30%]

(d) When integrating over the surface S which encloses a volume V, show that

$$\int_S n_x \, dS = \int_S n_y \, dS = \int_S n_z \, dS = 0$$

where $\boldsymbol{n} = (n_x, n_y, n_z)$ is the unit outward normal vector on the boundary of V. [20%]

2 The function u minimises the functional

$$I = \frac{1}{2}\int_V \nabla u \cdot \nabla u \, dV + \frac{1}{2}\int_V ku^2 \, dV - \int_V fu \, dV$$

where the constant $k > 0$ and the function f are given.

(a) Find the weak form of the differential equation which is satisfied by u. [20%]

(b) Find the strong form of the differential equation which is satisfied by u. [20%]

(c) If we wish to impose the boundary condition $\nabla u \cdot \boldsymbol{n} = h$ on part of the boundary of V, denoted by S_h, how should I be modified to satisfy this condition? Prove that minimising the modified functional will satisfy this boundary condition. [50%]

(d) Comment on the sign of k in the context of minimising I. [10%]

3 (a) If $\nabla^2 u = 0$ in a region V surrounded by a closed surface S, show that u is given by

$$u(\underline{x}_0) = \int_S u(\underline{x}) \frac{\partial G(\underline{x}, \underline{x}_0)}{\partial n} dS - \int_S G(\underline{x}, \underline{x}_0) \frac{\partial u(\underline{x})}{\partial n} dS$$

where $G(\underline{x}, \underline{x}_0)$ is a Green's Function.

Explain how this representation theorem is used to derive solutions for u which satisfy the boundary condition $u = U(\underline{x})$ for $\underline{x}$ on the boundary S. [25%]

(b) Show that the Green's Function for the region inside the sphere of radius a centred on the origin, shown in Fig. 1, is given by

$$G(\underline{x}, \underline{x}_0) = -\frac{1}{4\pi|\underline{x} - \underline{x}_0|} + \frac{a}{|\underline{x}_0|} \frac{1}{4\pi|\underline{x} - \underline{x}_1|}$$

where $\underline{x}_1$ is on the same radial line as $\underline{x}_0$ and where $|\underline{x}_1| = \dfrac{a^2}{|\underline{x}_0|}$. [25%]

(c) By differentiating $|\underline{x} - \underline{x}_0|^2 = r^2 + r_0^2 - 2 r r_0 \cos\alpha$, and a similar expression for $|\underline{x} - \underline{x}_1|^2$, show that, on $r = a$,

$$\frac{\partial}{\partial r}|\underline{x} - \underline{x}_0| = \frac{a - r_0 \cos\alpha}{|\underline{x} - \underline{x}_0|} \quad \text{and} \quad \frac{\partial}{\partial r}|\underline{x} - \underline{x}_1| = \frac{r_0 - a\cos\alpha}{|\underline{x} - \underline{x}_0|} .$$ [20%]

(d) Show that

$$u(\underline{x}_0) = \frac{a^2 - |\underline{x}_0|^2}{4\pi a} \int_{|\underline{x}|=a} \frac{U(\underline{x})}{|\underline{x} - \underline{x}_0|^3} dS$$ [30%]

(contd.

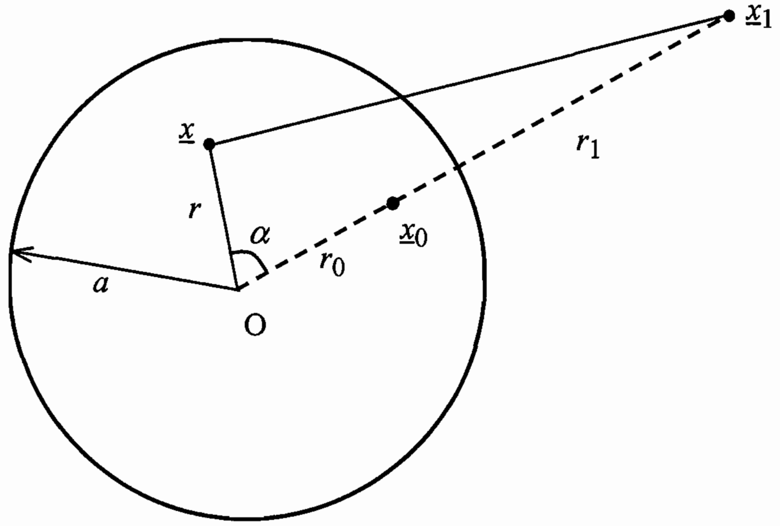

Fig. 1

(TURN OVER

4 (a) Explain what is meant by the method of characteristics and describe its relevance to the solution of second order partial differential equations. [15%]

(b) The function $u(x,y)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 u}{\partial x \partial y} - \frac{1}{x}\frac{\partial u}{\partial x} = -x^2 \qquad (1)$$

for $x \geq 1$ and for all y.

Show that $\xi = y - \dfrac{x^2}{2}$ and $\eta = y$ represent characteristic variables for equation (1). [25%]

(c) Find the general solution of equation (1) and hence solve it subject to the boundary conditions

$$u(1,y) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(1,y) = -y-1$$ [40%]

(d) Explain carefully why it is not appropriate to solve equation (1) in the domain $x \geq 0$ with boundary conditions applied on $x = 0$. [20%]

End of Paper

Engineering Tripos Part IIA and Part IIB
Module 4M12: Partial Differential Equations and Variational Methods

Data Sheet

1 Index notation

1. $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

2. $\epsilon_{ijk} = \begin{cases} 0 & \text{if any two of } i,j,k \text{ are equal} \\ 1 & \text{if } (i,j,k) \text{ is a permutation of } (123) \\ -1 & \text{if } (i,j,k) \text{ is a permutation of } (321) \end{cases}$

3. $[\boldsymbol{x} \times \boldsymbol{y}]_i = \epsilon_{ijk} x_j y_k$

4. $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

5. $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}$

6. $[\text{grad}\,\phi]_i = [\nabla\phi]_i = \dfrac{\partial \phi}{\partial x_i}$, $\quad \text{div}\,\boldsymbol{u} = \nabla \cdot \boldsymbol{u} = \dfrac{\partial u_i}{\partial x_i}$, $\quad [\text{curl}\,\boldsymbol{u}]_i = [\nabla \times \boldsymbol{u}]_i = \epsilon_{ijk}\dfrac{\partial u_k}{\partial x_j}$

2 Integral theorems

1. Divergence theorem:

$$\int_V \frac{\partial}{\partial x_i}(\cdot)\, dV = \int_S (\cdot) n_i\, dS$$

where V is a volume enclosed by the surface S, $(\cdot)$ is any permissible index notation expression, and $\boldsymbol{n}$ is the unit outward normal vector to the volume V.

2. Stokes's theorem:

$$\int_S \epsilon_{ijk} \frac{\partial}{\partial x_j}(\cdot) n_i\, dS = \oint_C (\cdot) s_k\, dC$$

where S is surface (possible curved) with a curve C running around the boundary of S, $\boldsymbol{n}$ is the unit normal vector to the surface S, $(\cdot)$ is any permissible index notation expression and $\boldsymbol{s}$ is the unit vector tangential to the edge of S.

3 Variational methods

1. To minimise $I = \int_0^L F(y, y', x)\, dx$, F must satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

at all x.

2. For the above problem, if F depends on y' but not on y, then

$$\frac{\partial F}{\partial y'} = k, \quad \text{where } k \text{ is a constant.}$$

If F does not depend explicitly on x (i.e., only depends on x via y and y'), then

$$F - y'\frac{\partial F}{\partial y'} = k, \quad \text{where } k \text{ is a constant.}$$

3. The directional derivative of the functional $f(\boldsymbol{u})$ is given by

$$Df(\boldsymbol{u})[\boldsymbol{v}] = \left.\frac{df(\boldsymbol{u} + \epsilon \boldsymbol{v})}{d\epsilon}\right|_{\epsilon=0}$$

4 Partial differential equations

1. Classification: The second-order quasi-linear partial differential equation

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

is:

	hyperbolic	where $b^2 - ac > 0$
is:	*parabolic*	where $b^2 - ac = 0$
	elliptic	where $b^2 - ac < 0$

2. Well-posed problem: A problem is well-posed if the solution
   a) exists
   b) is unique
   c) depends continuously on the input data (i.e. is stable with respect to changes in the input data)

3. Common reference equations are:

Helmholtz equation	$\nabla^2 u + k^2 u = 0$
Poisson equation	$\nabla^2 u = f(x, y)$
Laplace equation	$\nabla^2 u = 0$
Wave equation	$\dfrac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = 0$
Diffusion equation	$\dfrac{\partial u}{\partial t} - \alpha \nabla^2 u = 0$

The form of the Laplacian operator ∇^2 in various coordinate systems can be found in the Maths Data Book.

4. D'Alembert travelling wave solution: the solution of

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial t^2} \quad \text{for } t > 0 \text{ and for all } x$$

with the initial conditions $u(x,0) = \phi(x)$ and $\partial u(x,0)/\partial t = \psi(x)$ is

$$u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\, d\xi.$$

5. Fundamental solution (free-space Green's function):

2D Poisson/Laplace equation

$$\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0), \quad G(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{1}{2\pi} \ln|\boldsymbol{x} - \boldsymbol{x}_0|$$

3D Poisson/Laplace equation

$$\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}_0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0), \quad G(\boldsymbol{x}, \boldsymbol{x}_0) = -\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|}$$

Fundamental solution:

Diffusion equation

$$\frac{\partial F}{\partial t} - \alpha \frac{\partial^2 F}{\partial x^2} = \delta(x - x_0)\delta(t - t_0)$$

$$F(x, t; x_0, t_0) = \frac{1}{\sqrt{4\alpha\pi(t - t_0)}} \exp\left(-\frac{(x - x_0)^2}{4\alpha(t - t_0)}\right) \quad \text{for } t > t_0$$

3-(space)D wave equation

$$\frac{\partial^2 F}{\partial t^2} - c^2 \nabla^2 F = \delta(t - t_0)\delta(\boldsymbol{x} - \boldsymbol{x}_0)$$

$$F(\boldsymbol{x}, t; \boldsymbol{x}_0, t_0) = \frac{\delta\left(t - t_0 - \dfrac{|\boldsymbol{x} - \boldsymbol{x}_0|}{c}\right)}{4\pi c^2 |\boldsymbol{x} - \boldsymbol{x}_0|} \quad \text{for } t > t_0$$