ENGINEERING TRIPOS PART IIB
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Tuesday 5 May 2009 2:30 to 4

Module 4M13

COMPLEX ANALYSIS AND OPTIMIZATION

*Answer not more than **three** questions.*

The questions may be taken from any section.

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:

4M13 Datasheet (4 pages).

Answers to Sections A and B should be tied together and handed in separately.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Single-sided script paper	Engineering Data Book
Graph paper	CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1 (a) Identify the *isolated* singularities of the following complex functions, and calculate the residue at each one.

(i) $f(z) = 1/(z^3 - z^5)$ [25%]

(ii) $f(z) = 1/\sin(1/z)$ [25%]

(b) Using contour integration, evaluate the following integral,

$$\int_0^{2\pi} \cos^4 x + \sin^4 x \, dx$$ [50%]

2 (a) State *Jordan's Lemma*. [10%]

(b) Evaluate, using contour integration,

$$\int_0^\infty \frac{\cos mx}{x^2 + a^2}\,dx,$$

where $a > 0$. [90%]

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SECTION B

3 A nuclear engineer has been asked to design a flask to contain a certain volume V of some highly radioactive nuclear waste. In order to shield workers adequately, the walls of the flask must be of a certain thickness t. The engineer has decided to make the flask cylindrical. The interior cavity in which the waste will be stored will be of internal radius R and height H, as indicated schematically in Fig. 1.

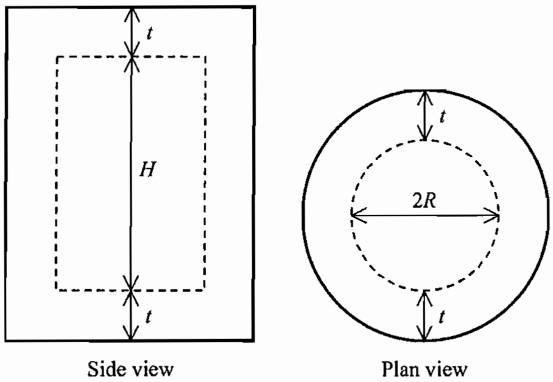

Fig. 1

In order to provide effective shielding, the material from which the flask will be made is very dense. The engineer therefore decides to minimize the volume of material used in the construction of the flask. As t is defined, the design variables are R and H.

(a) Formulate this task as a constrained optimization problem with one equality constraint. [10%]

(b) By using the equality constraint to eliminate H from your expression for the objective function, show that the task can be formulated as an unconstrained univariate minimization problem with an objective function

$$f(R) = 2\pi(R+t)^2 t + V\left[\left(1+\frac{t}{R}\right)^2 - 1\right]$$ [10%]

(c) Estimate, using a Golden Section line search, the value of R that minimizes f for the case where $V = 1000\pi$ cm^3 and $t = 10$ cm. A suitable initial interval for R is

(cont.

between 5 and 10 cm, and the search can be halted when the interval has been reduced four times. [40%]

(d) By using appropriate optimality criteria find an analytical expression in terms of V and t for the value of R that minimizes f. Hence find the optimal flask design for the case detailed in (c), and comment on the performance of the Golden Section line search. How does the optimal flask design for a given value of V change with the wall thickness t? [40%]

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4 An engineer has been asked to redesign a parallel flow shell-and-tube heat exchanger to improve its heat transfer performance. A schematic end view of the heat exchanger is shown in Fig. 2.

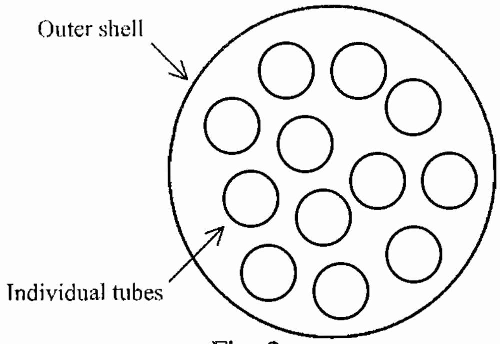


Fig. 2

The smallest available conducting tube has a radius of 0.5 cm and all the tubes used must be of the same size. To ensure there is adequate space inside the outer shell, the total cross-sectional area occupied by the tubes must not exceed 100 cm^2. The heat transfer performance of the heat exchanger depends directly on the total surface area of the tubes. The tubes are of fixed length l.

(a) Formulate the task of optimizing the design of the heat exchanger as a constrained *minimization* problem in standard form. [10%]

(b) Treating the number of tubes N as a continuous variable, identify the feasible region graphically. By superimposing contours of the objective function, identify which of the constraints are active at the optimum, and hence find the optimal solution to this design problem. [30%]

(c) Confirm your results to (b) by solving the same minimization problem using the Kuhn-Tucker multiplier method. [50%]

(d) Briefly discuss the merits of trying to change the limiting values of the constraints. [10%]

END OF PAPER

4M13

OPTIMIZATION

DATA SHEET

1. **Taylor Series Expansion**

For one variable:

$$f(x) = f(x^*) + (x - x^*)f'(x^*) + \frac{1}{2}(x - x^*)^2 f''(x^*) + R$$

For several variables:

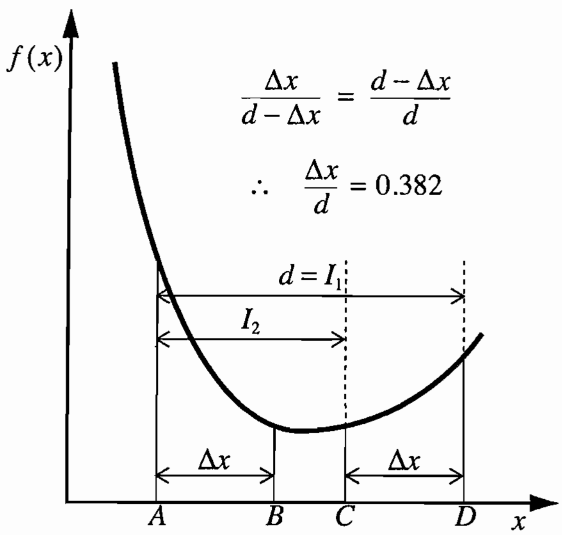
$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \boldsymbol{H}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) + R$$

where

$$\text{gradient } \nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \quad \text{and hessian } \boldsymbol{H}(\mathbf{x}) = \nabla(\nabla f(\mathbf{x})) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$\boldsymbol{H}(\mathbf{x}^*)$ is a symmetric $n \times n$ matrix and R includes all higher order terms.

2. **Golden Section Method**



(a) Evaluate $f(x)$ at points A, B, C and D.

(b) If $f(B) < f(C)$, new interval is $A - C$.

If $f(B) > f(C)$, new interval is $B - D$.
If $f(B) = f(C)$, new interval is either $A - C$ or $B - D$.

(c) Evaluate $f(x)$ at new interior point. If not converged, go to (b).

3. **Newton's Method**

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -H(\mathbf{x}_k)^{-1}\nabla f(\mathbf{x}_k)$

(c) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k$

(d) Test for convergence. If not converged, go to step (b)

4. **Steepest Descent Method**

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -\nabla f(\mathbf{x}_k)$

(c) Perform line search to determine step size α_k or evaluate $\alpha_k = \dfrac{\mathbf{d}_k^T\mathbf{d}_k}{\mathbf{d}_k^T H(\mathbf{x}_k)\mathbf{d}_k}$

(d) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k\mathbf{d}_k$

(e) Test for convergence. If not converged, go to step (b)

5. **Conjugate Gradient Method**

(a) Select starting point $\mathbf{x}_0$ and compute $\mathbf{d}_0 = -\nabla f(\mathbf{x}_0)$ and $\alpha_0 = \dfrac{\mathbf{d}_0^T\mathbf{d}_0}{\mathbf{d}_0^T H(\mathbf{x}_0)\,\mathbf{d}_0}$

(b) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k\mathbf{d}_k$

(c) Evaluate $\nabla f(\mathbf{x}_{k+1})$ and $\beta_k = \left[\dfrac{|\nabla f(\mathbf{x}_{k+1})|}{|\nabla f(\mathbf{x}_k)|}\right]^2$

(d) Determine search direction $\mathbf{d}_{k+1} = -\nabla f(\mathbf{x}_{k+1}) + \beta_k\mathbf{d}_k$

(e) Determine step size $\alpha_{k+1} = -\dfrac{\mathbf{d}_{k+1}^T\nabla f(\mathbf{x}_{k+1})}{\mathbf{d}_{k+1}^T H(\mathbf{x}_{k+1})\,\mathbf{d}_{k+1}}$

(f) Test for convergence. If not converged, go to step (b)

6. **Gauss-Newton Method (for Nonlinear Least Squares)**

If the minimum squared error of residuals $\mathbf{r}(\mathbf{x})$ is sought:

$$\text{Minimise } f(\mathbf{x}) = \sum_{i=1}^{m} r_i^2(\mathbf{x}) = \mathbf{r}(\mathbf{x})^T\mathbf{r}(\mathbf{x})$$

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -[\,J(\mathbf{x}_k)^T J(\mathbf{x}_k)\,]^{-1}\,J(\mathbf{x}_k)^T\mathbf{r}(\mathbf{x}_k)$

$$\text{where } \boldsymbol{J}(\mathbf{x}) = \begin{bmatrix} \nabla r_1(\mathbf{x})^T \\ \vdots \\ \nabla r_m(\mathbf{x})^T \end{bmatrix} = \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_m}{\partial x_1} & \cdots & \frac{\partial r_m}{\partial x_n} \end{bmatrix}$$

(c) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k$

(d) Test for convergence. If not converged, go to step (b)

7. **Lagrange Multipliers**

To minimise $f(\mathbf{x})$ subject to m equality constraints $h_i(\mathbf{x}) = 0, i = 1, \ldots, m$, solve the system of simultaneous equations

$$\begin{aligned} \nabla f(\mathbf{x}^*) + [\nabla \mathbf{h}(\mathbf{x}^*)]^T \boldsymbol{\lambda} &= 0 \quad (n \text{ equations}) \\ \mathbf{h}(\mathbf{x}^*) &= 0 \quad (m \text{ equations}) \end{aligned}$$

where $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_m]^T$ is the vector of Lagrange multipliers and

$$[\nabla \mathbf{h}(\mathbf{x}^*)]^T = \left[\nabla h_1(\mathbf{x}^*) \ \ldots \ \nabla h_m(\mathbf{x}^*) \right] = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_1}{\partial x_n} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$$

8. **Kuhn-Tucker Multipliers**

To minimise $f(\mathbf{x})$ subject to m equality constraints $h_i(\mathbf{x}) = 0, i = 1, \ldots, m$ and p inequality constraints $g_i(\mathbf{x}) \leq 0, i = 1, \ldots, p$, solve the system of simultaneous equations

$$\begin{aligned} \nabla f(\mathbf{x}^*) + [\nabla \mathbf{h}(\mathbf{x}^*)]^T \boldsymbol{\lambda} + [\nabla \mathbf{g}(\mathbf{x}^*)]^T \boldsymbol{\mu} &= 0 \quad (n \text{ equations}) \\ \mathbf{h}(\mathbf{x}^*) &= 0 \quad (m \text{ equations}) \\ \forall\ i = 1, \ldots, p, \quad \mu_i g_i(\mathbf{x}) &= 0 \quad (p \text{ equations}) \end{aligned}$$

where $\boldsymbol{\lambda}$ are Lagrange multipliers and $\boldsymbol{\mu} \geq 0$ are the Kuhn-Tucker multipliers.

9. **Penalty & Barrier Functions**

To minimise $f(\mathbf{x})$ subject to p inequality constraints $g_i(\mathbf{x}) \le 0$, $i = 1, \ldots, p$, define

$$q(\mathbf{x}, p_k) = f(\mathbf{x}) + p_k P(\mathbf{x})$$

where $P(\mathbf{x})$ is a penalty function, e.g.

$$P(\mathbf{x}) = \sum_{i=1}^{p} (\max[0, g_i(\mathbf{x})])^2$$

or alternatively

$$q(\mathbf{x}, p_k) = f(\mathbf{x}) - \frac{1}{p_k} B(\mathbf{x})$$

where $B(\mathbf{x})$ is a barrier function, e.g.

$$B(\mathbf{x}) = \sum_{i=1}^{p} \frac{1}{g_i(\mathbf{x})}$$

Then for successive $k = 1, 2, \ldots$ and p_k such that $p_k > 0$ and $p_{k+1} > p_k$, solve the problem

$$\text{minimise } q(\mathbf{x}, p_k)$$