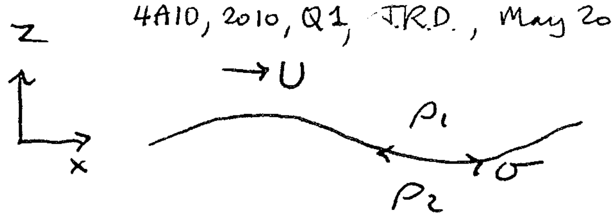
4A10, 2010, Q1, J.R.D., May 2010

Q1.



$z = \eta(x,t) = \eta_0 e^{st+ikx}$

a)

V. potentials:

Top flow:

$$\phi_1 = Ux + Ae^{st+ikx}e^{-kz}$$

Bottom:

$$\phi_2 = Be^{st+ikx}e^{kz}$$

$$\left.\frac{\partial \phi_1}{\partial z}\right|_{z=0} = \eta\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)$$

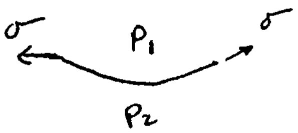
$$-kAe^{st+ikx-kz} = \eta_0(s+ikU)e^{st+ikx+kz}$$

$$-kA = (s+ikU)\eta_0$$

$$\left.\frac{\partial \phi_2}{\partial z}\right|_{z=0^-} = \frac{\partial \eta}{\partial t}$$

$$kBe^{st+ikx+kz} = s\eta_0 e^{st+ikx}$$

$$kB = s\eta_0 \quad \text{(ignoring small terms)}$$



$$P_1 = P_2 + \sigma\frac{\partial^2 \eta}{\partial x^2}$$

$$\therefore P_1 = P_2 - \sigma k^2 \eta_0$$

Top $$\frac{P_1}{\rho_1} = -\frac{\partial \phi_1}{\partial t} - U\frac{\partial \phi_1}{\partial x} = -A(s+ikU)e^{st+ikx}\Big|_{z=0^+}$$

Bottom $$\frac{P_1}{\rho_2} = -\frac{\partial \phi_2}{\partial t} = -sBe^{st+ikx}$$

$$-\sigma k^2 \eta_0 = P_1 - P_2 = -\rho_1(s+ikU)Ae^{st+ikx} + \rho_2 sBe^{st-ikx} \quad \text{— (1)}$$

sub A:B into (1)

$$A = -\frac{\eta_0(s+ikU)}{k}, \quad B = \frac{s\eta_0}{k}$$

$$-\sigma k^3 = \rho_1(s+ikU)^2 + \rho_2 s^2$$

$$\therefore \rho_1(s+ikU)^2 + \rho_2 s^2 - \sigma k^3 = 0$$

re-arrange: $$s^2(\rho_1+\rho_2) + 2\rho_1 sikU - \rho_1 k^2U^2 + \sigma k^3 = 0$$

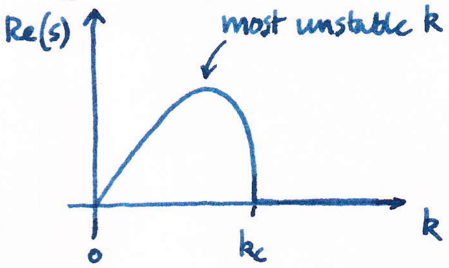
$$s^2 + \frac{2\rho_1 sikU}{\rho_1+\rho_2} - \frac{\rho_1 k^2U^2}{\rho_1+\rho_2} + \frac{\sigma k^3}{\rho_1+\rho_2} = 0$$

$$s = -\frac{\rho_1 ikU}{\rho_1+\rho_2} \pm \left[\frac{\rho_1\rho_2 k^2U^2}{(\rho_1+\rho_2)^2} - \frac{\sigma k^3}{\rho_1+\rho_2}\right]^{1/2}$$

$$s = -\frac{\rho_1 ikU}{\rho_1+\rho_2} \pm \left(\frac{k}{\rho_1+\rho_2}\right)^{1/2}\left(\frac{\rho_1\rho_2 kU^2}{\rho_1+\rho_2} - \sigma k^2\right)^{1/2}$$

[60%]

b) Find the most unstable k. This question was written before spatio-temporal stability analysis was in the course so it assumes implicitly that the stability analysis is temporal. Therefore k is real and we need to search for the maximum value of Real(s), which is the growth rate:



From part (a) we know that:

$$s = -ik\frac{\rho_1}{\rho_1+\rho_2}U \pm \left[ak^2 - bk^3\right]^{1/2}$$

where $a \equiv \frac{\rho_1\rho_2 U^2}{(\rho_1+\rho_2)^2}$ and $b \equiv \frac{\sigma}{\rho_1+\rho_2}$

s is a pure imaginary number unless $ak^2 - bk^3 > 0$ so we will restrict our search to values of k that satisfy $ak^2 - bk^3 > 0$. (1)

$\text{Real}(s) = \pm\left[ak^2 - bk^3\right]^{1/2}$; the max. of Real(s) occurs where $\frac{d}{dk}\text{Real}(s) = 0$

$$\frac{d}{dk}\left[ak^2 - bk^3\right]^{1/2} = \frac{1}{2}\left[ak^2 - bk^3\right]^{-1/2}(2ak - 3bk^2) = 0$$

This is satisfied when $k = 0$, $k = \infty$, or ~~2ak~~ $2a - 3bk = 0$. The first two values of k violate (1) so we require $k = \frac{2a}{3b} = \frac{2}{3}\frac{\rho_1\rho_2 U^2}{\cancel{\sigma}(\rho_1+\rho_2)}$ [20%]

c) i) $\frac{\sigma}{L}$ is the surface tension force per unit area

ρU^2 is momentum flux / dynamic pressure

$\therefore$ We = $\frac{\text{momentum flux}}{\text{surface tension force}}$ [10%]

ii) as $\rho_2 \gg \rho_1$ relative momentum flux ratio would have a strong influence. [10%]

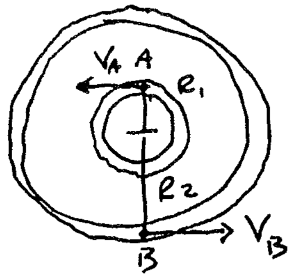
Examiners' comments:

(a) Few candidates managed to get full marks for this part. Some approached the problem as Rayleigh break-up of a liquid jet rather than a shear flow instability. Most marks were lost on algebra, when substituting the kinematic terms into the pressure balance, rather than a poor understanding on how to solve the problem.

(b) Most didn't realise that differentiation was required to get the most unstable wave number.

(c) Most candidates understood the physical meaning of We. No candidates realised that the momentum ratio was key when the fuel film was moving, but a few almost got there by stating that the velocity ratio was important. Almost everyone suggested Re.

Q2. i)



2 rings of fluid at $r = A$ & B of mass δm

$$KE = \frac{1}{2}\delta m\,(V_1^2 + V_2^2)$$

Create an axisymmetric perturbation by displacing ring A to position of ring B.

Cons of ang. mom $\quad r_1 V_1 = r_2 V_{new}$

$$V_{new} = \frac{r_1}{r_2} V_1$$

By continuity ring B moves to A $\quad V_{new} = \frac{r_2}{r_1} V_2$

$$KE = \frac{1}{2}\delta m\left(\frac{r_1^2 V_1^2}{r_2^2} + \frac{r_2^2 V_2^2}{r_1^2}\right)$$

$$\begin{aligned}
\Delta KE &= KE_{Perturbed} - KE_{FINAL} \\
&= \frac{1}{2}\delta m\left[(V_1^2 + V_2^2) - \left(\frac{r_1^2 V_1^2}{r_2^2} + \frac{r_2^2 V_2^2}{r_1^2}\right)\right] \\
&= \frac{1}{2}\delta m\left[r_1^2 V_1^2\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) + r_2^2 V_2^2\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right)\right] \\
&= \frac{1}{2}\delta m\,(r_1^2 V_1^2 - r_2^2 V_2^2)\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)
\end{aligned}$$

[20%]

$$\Gamma = 2\pi r V \qquad = \frac{\delta m}{8\pi^2}(\Gamma_1^2 - \Gamma_2^2)\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

$\Gamma_1^2 > \Gamma_2^2$ KE increases

$\Gamma_2^2 > \Gamma_1^2$ KE decreases

∴ For an inviscid flow if Γ^2 increases with r flow is STABLE. If Γ^2 decreases with r flow is UNSTABLE

[10%]

iii) Centrifugal instability caused by an imbalance between radial p-gradient and centripetal force so fluid moves radially outwards.

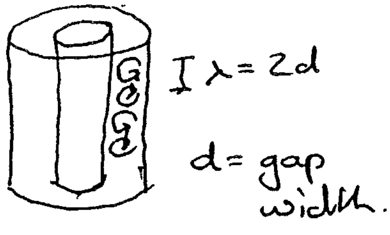
If outer cylinder is fixed, and inner cylinder starts rotating faster flow will reach a critical pt when Γ^2 decreases with r.

[20%]

[cont.

The flow pattern depends on $Re = \frac{\Omega r_1}{\nu}$ and there are several critical values of Re.

Above the first critical value, flow breaks down into STEADY toroidal vortices whose pattern repeats with $\lambda = 2d$, each vortex having an alternate rotational direction.



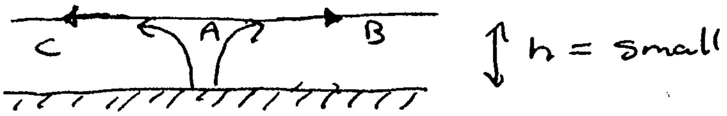
Above the next critical Re, azimuthal waves are set-up on top of vortices and as Ω_1 is increased further flow can become chaotic.

At large Re flow becomes periodic (rolls with $\lambda = 2d$) and STEADY but TURBULENT.

b) i) Two types of instability resulting from a) buoyancy and b) variation in surface tension σ with temp.

a) buoyancy creates an unstable ρ-stratification resulting in convection rolls known as Rayleigh–Bernard instability.

b) if liquid height is small (i.e. thin) surface tension effect dominates resulting in a cellular instability – Marangoni. (when viewed normal to free-surface)



At A surface tension is reduced due to fluid being hotter that at B & C. Surface traction at B & C pulls fluid that upwells at A. A is centre of cell and flow moves down sides of cell when fluid is cooled.

[20%]

ii) Relevant non-dim $Ra = \frac{g\alpha \Delta T d^3}{\nu K}$, $Ma = \frac{-d\sigma}{dT}\frac{(T_0 - T_1)d}{\rho \nu K}$

[20%]

iii) depends on height of film / fluid and rate of heat input on non-dims.

when $d > \left[\left(\frac{-d\sigma}{dT}\right)\frac{Ra}{Ma}\frac{1}{\rho g k}\right]^{1/2} \Rightarrow$ Rayleigh–Bernard dominates.

[10%]

if h is small Ma will dominate.

JRD

Examiners' comments:

Both parts were bookwork questions and on the whole was well answered.

(a) This part of the question was well answered by almost all the candidates. Almost all candidates described the flow patterns correctly but more than half failed to state that the instability mechanism is due to a pressure imbalance.

(b) This part of the question was also well answered by most candidates.

(a) $f_1 = -ky_1 + k(y_2 - y_1) = m\ddot{y}_1$
$f_2 = -ky_2 - k(y_2 - y_1) = m\ddot{y}_2$ } Equations of motion for mass 1 & mass 2.

At resonance, assume simple harmonic motion of the form:

$$y_1 = y_{1_0} e^{i\omega t} \quad \text{and} \quad y_2 = y_{2_0} e^{i\omega t} \quad (\text{same } \omega)$$

substitute into equations of motion and re-arrange as a matrix:

$$\begin{bmatrix} m\omega^2 - 2k & k \\ k & m\omega^2 - 2k \end{bmatrix} \begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (1)$$

This has non-trivial solutions when the determinant of the matrix is zero.

$$\Rightarrow (m\omega^2 - 2k)^2 = k^2$$
$$\Rightarrow m\omega^2 - 2k = \pm k$$
$$\Rightarrow \omega^2 = \frac{3k}{m} \quad \text{and} \quad \omega^2 = \frac{k}{m}$$

These are the two resonant frequencies

To calculate the mode shapes, substitute the frequencies into (1)

$\omega^2 = \frac{3k}{m}$:
$$\begin{bmatrix} k & k \\ k & k \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
corresponding eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

ie. at this frequency, the masses move in opposite directions with the same magnitude.

$\omega^2 = k/m$:
$$\begin{bmatrix} -k & k \\ k & -k \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
corresponding eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

ie. at this frequency, the masses move in the same direction with the same magnitude. (This is equivalent to each mass being on a single spring so we expect $\omega^2 = k/m$) [20%]

[part (a) answered well by 1/2 candidates using a variety of techniques]

(b)

$$f_1 = -ky_1 + k(y_2 - y_1) - b\dot{y}_1 = m\ddot{y}_1$$
$$f_2 = -ky_2 - k(y_2 - y_1) - b\dot{y}_2 = m\ddot{y}_2$$

Equations of motion, including damping

Assume $y_1 = y_{1_0} e^{i\omega t}$ and $y = y_{2_0} e^{i\omega t}$ as before.

$$\Rightarrow \begin{bmatrix} m\omega^2 - bi\omega - 2k & k \\ k & m\omega^2 - bi\omega - 2k \end{bmatrix} \begin{bmatrix} y_{1_0} \\ y_{2_0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad m\omega^2 - bi\omega - 2k = \pm k$$

$\Rightarrow$ first solution is $\omega = \pm\left(3\dfrac{k}{m} - \dfrac{b^2}{4m^2}\right)^{1/2} + \dfrac{bi}{2m}$

second solution is $\omega = \pm\left(\dfrac{k}{m} - \dfrac{b^2}{4m^2}\right)^{1/2} + \dfrac{bi}{2m}$

with the same mode shapes as in part (a)

comments: if $b \to 0$, we recover the solution in part (a).

for $b > 0$, solution has positive imaginary part $\Rightarrow$ damping

" , real part is smaller than for $b = 0$ $\Rightarrow$ frequency reduces

for $\dfrac{b^2}{4m^2} > \dfrac{k}{m}$, there is no real part, $\Rightarrow$ no oscillations

[10%]

[part (b) was answered by very few candidates and only 1 out of 22 did the full calculation correctly. Most gave a qualitative description]

(c) When the cylinders are placed in a fluid, the fluid has to move around them as they oscillate. The corresponding inertial forces can be modelled by an added mass on each cylinder ($\rho\pi d^2/4$ for the circular cylinder and $1.51\rho\pi d^2/4$ for the square cylinder). The two resonant frequencies will decrease because k remains the same. The mode shapes will remain qualitatively the same but the circular cylinder will have larger amplitude.

(The fluid viscosity will cause viscous dissipation, which can be modelled as added damping. This will have the same effect as the damping described in part (b).)

From (a), $\omega \propto \left(\frac{k}{m}\right)^{1/2}$ which implies that $\frac{\delta\omega}{\omega} = -\frac{1}{2}\frac{\delta m}{m}$

$\Rightarrow \frac{\delta m}{m} < -2 * 0.01$ where δm is the added mass and m is the mass of the cylinder (per unit length)

$\Rightarrow \frac{\rho\pi d^2}{4} * \frac{L}{m} < -2 * 0.01$ (for the circular cylinder)

$\Rightarrow \rho < -0.02 \frac{m}{\frac{\pi d^2 L}{4}}$

$\Rightarrow \rho_{fluid} < -0.02\, \rho_{solid}$

The fluid will affect the resonant frequency by less than 1% if its density is less than 2% of that of the solid.

[20%]

[Most students gave a correct qualitative description of added mass. Around 1/2 performed the calculation. No student wrote $\delta\omega/\omega = -1/2\, \delta m/m$ but some found the same result in more elaborate ways.]

(d) The square cylinder can gallop. The fluid gives rise to a force $F = \frac{1}{2}\rho U^2 d\, C_y$, where C_y is shown in Fig b as a function of apparent angle of attack.

Taylor expanding C_y and noting that $\alpha \simeq \dot{y}/U$ leads to:

$$C_y \simeq \frac{\dot{y}}{U}\left.\frac{\partial C_y}{\partial \alpha}\right|_{\alpha=0} \quad \text{for small } \dot{y}/U.$$

Substituting into the governing equation for cylinder 1 gives:

$$f_1 = -ky_1 + k(y_2 - y_1) - \left(b - \frac{1}{2}\rho U d \left.\frac{\partial C_y}{\partial \alpha}\right|_{\alpha=0}\right)\dot{y}_1 = m\ddot{y}_1$$

This has negative damping when $\frac{1}{2}\rho U d \left.\frac{\partial C_y}{\partial \alpha}\right|_{\alpha=0} > b$

and will start to gallop when $U > \dfrac{b}{\frac{1}{2}\rho d\, \partial C_y/\partial\alpha|_{\alpha=0}}$. [20%]

[many students regurgitated the notes, which was 4 easy marks for those who knew gallop well. Long descriptions of the gallop mechanism were unnecessary]

(e) The system is stable to soft excitation but, from the shape of $C_y(\alpha)$, we can see that it is potentially unstable to hard excitation. We keep this in mind.

Both cylinders will shed vortices at $Re \sim 10^4$, the circular cylinder at $St \simeq 0.2$ and the square cylinder at $St \simeq 0.13$. The system is lightly damped, so $\omega_1 \simeq (3k/m)^{1/2}$ and $\omega_2 \simeq (k/m)^{1/2}$ are the angular frequencies at which the forcing from vortex shedding will cause the greatest response in the system. (The system must be lightly damped because the velocity is just below the galloping velocity of the square cylinder, which means that the overall damping is approximates zero)

In general, the system will be particularly unstable when $\omega = 2\pi f = 2\pi\, St \dfrac{U}{d} \Rightarrow d = 2\pi U \dfrac{St}{\omega}$. There will be four such frequencies, corresponding to the four combinations of ω_1, ω_2 and St_1, St_2. [30%]

[many students regurgitated notes on vortex shedding without thinking about the potential interaction with the other cylinder. Only a few attempted to derive the particularly unstable frequencies]

Examiners' comments:

(a) This part was answered well by around half the candidates using a variety of techniques. Applying symmetry in order to write down the resonant frequencies was a particularly elegant technique, if successful, but caused some candidates to fail spectacularly. The matrix technique in the crib can be found in 1A maths and is the safest.
(b) Most candidates answered this part qualitatively but did not attempt the calculation.
(c) Most candidates gave a correct qualitative description of added mass and around half performed the calculation. Nobody found the easy way to do the calculation but a handful derived the correct result.
(d) This was a bookwork question and many candidates regurgitated the notes successfully, showing that they understood the mechanism of gallop and could derive the critical velocity. Most of these candidates used the nomenclature in the notes, rather than that in the question, but this was not penalized.
(e) This was not a bookwork question, although many candidates regurgitated the notes on vortex shedding unnecessarily. Most candidates mentioned lock-in between the vortex shedding frequency and the natural resonant frequency of the system but few mentioned the potential for hard excitation of the square cylinder. Only a handful attempted to quantify the frequencies at which the system will be particularly unstable.

4A10, 2010, Q4, M.P.J., May 2010

(a) $$m\ddot{x} + 2m\zeta\omega_n\dot{x} + kx = F$$

consider forcing $F = F_0 e^{i\omega t}$ and the response at frequency ω: $x_0 e^{i\omega t}$

$$\Rightarrow -m\omega^2 x_0 + 2m\zeta\omega_n i\omega x_0 + kx_0 = F_0$$

$$\Rightarrow x_0 = \frac{F_0}{(k - m\omega^2) + 2m\zeta\omega_n i\omega}$$ [10%]

(b) $$\left|\frac{x_0}{F_0}\right| = \left|\frac{1}{(k-m\omega^2) + 2m\zeta\omega_n i\omega}\right|$$

n.b. $\left|\frac{1}{z}\right| = \frac{1}{|z|}$; proof: $\left|\frac{1}{z}\right| = \left|\frac{z^*}{zz^*}\right| = \left|\frac{z^*}{|z|^2}\right| = \frac{|z^*|}{|z|^2} = \frac{|z|}{|z|^2} = \frac{1}{|z|}$

$$\Rightarrow |H_{Fx}(\omega)|^2 = \left|\frac{x_0}{F_0}\right|^2 = \frac{1}{(k-m\omega^2)^2 + (2m\zeta\omega_n\omega)^2}$$

but $\omega_n^2 = k/m$ so $$= \frac{1/m^2}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

$$\Rightarrow |H_{Fx}(\omega)|^2 = \frac{1}{m^2\omega_n^4}\,\underbrace{\frac{1}{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}} \quad (1)$$

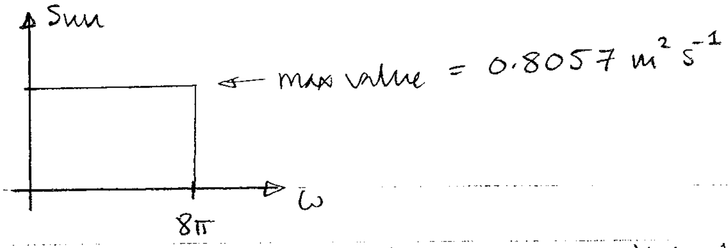
[These parts were answered well. A common mistake was leaving k in the expression, while the question specifically asks for m, ω_n, ζ and ω.]

note that this is sketched in the mechanics data book as Y/X [10%]

(c) $$\int_0^\infty S_{uu}\,d\omega = \overline{u'(t)^2} = (0.15 * 30)^2\ m^2 s^{-2}$$ by definition of S_{uu}.

$$\Rightarrow S_{uu\,max} * 8\pi\ s^{-1} = (4.5)^2\ m^2 s^{-2}$$

$$\Rightarrow S_{uu\,max} = \frac{4.5^2}{8\pi} = 0.8057\ m^2 s^{-1}$$



[20%]

[This was answered reasonably well, although many candidates did not divide by 8π.]

(d) $F = C_D \frac{1}{2} \rho U^2 A$

If $F = \bar{F} + F'$ and $U = \bar{U} + u'$

then $\bar{F} + F' = C_D \frac{1}{2} \rho A (\bar{U}^2 + 2\bar{U}u' + u'^2)$ ← assume small and ∴ neglect.

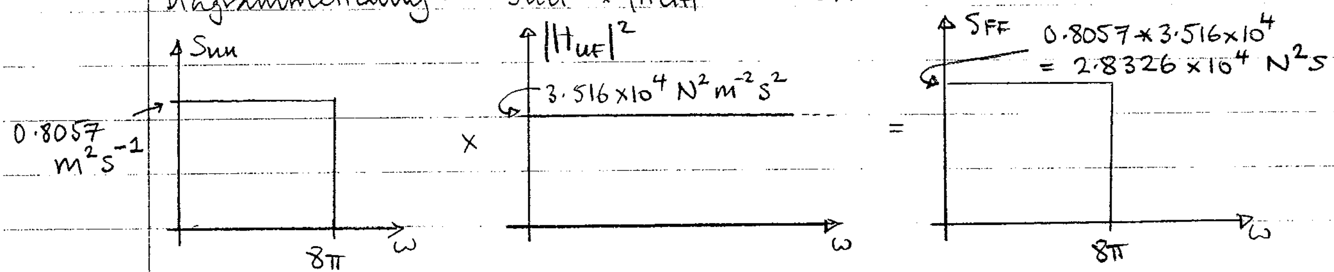
$\Rightarrow \quad \bar{F} = C_D \frac{1}{2} \rho \bar{U}^2 A \quad$ and $\quad F' = C_D \rho \bar{U} u' A$

$$\Rightarrow |H_{uF}|^2 \equiv \left|\frac{F'}{u'}\right|^2 = (C_D \rho \bar{U} A)^2 = (1 * 1.25 * 30 * 5)^2$$
$$= 3.516 \times 10^4 \ N^2 m^{-2} s^2$$

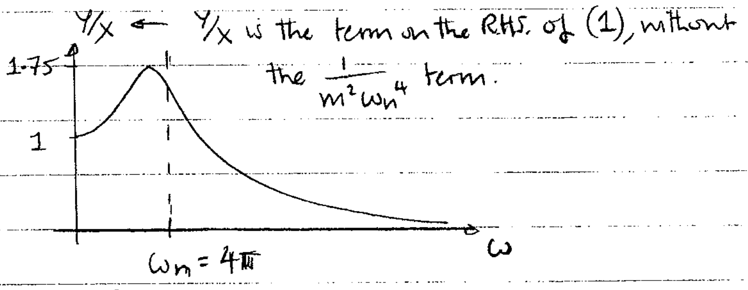
This does not depend on ω. [20%]

(e) $S_{xx}(\omega) = |H_{Fx}|^2 * |H_{uF}|^2 * S_{uu}$

Diagrammatically: $S_{uu} * |H_{uF}|^2 = S_{FF}$:



From the mechanics data book with $\zeta = 0.3$:

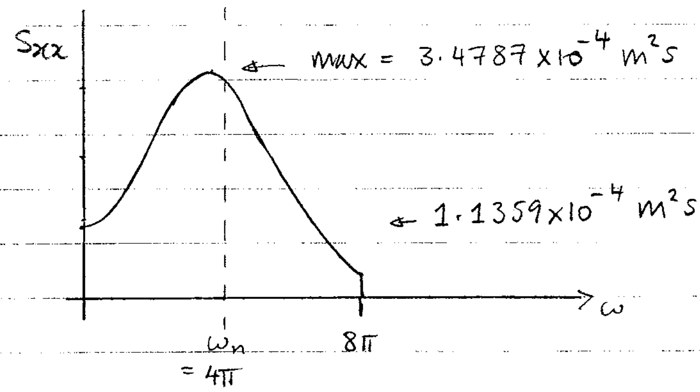


$$|H_{Fx}|^2_{max} = \frac{1}{m^2 \omega_n^4} * 1.75^2$$

$$\Rightarrow S_{xx\,max} = 1.75^2 * \frac{1}{100^2 * (4\pi)^4} * 2.8326 \times 10^4 \quad \frac{N^2 s}{kg^2 s^{-4}}$$

$$= 3.4787 \times 10^{-4} \, m^2 s$$

$\Rightarrow$



[40%]

spectrum shifts to right if ω_n increased. (If it shifts far enough, the dish is less susceptible to turbulent buffeting). $|H_{Fx}(\omega)|^2$ also reduces strongly, also reducing buffet.

[Most candidates showed that they understood the principle, drawing the diagram of S_{uu}, $|H_{uF}|^2$, S_{FF} and S_{xx} from the notes. Most managed to draw $S_{xx}(\omega)$ but few put numbers onto the axes, which was the main part of the question. Most candidates correctly answered the final part of the question]

Examiners' comments:

(a-b) These parts were answered well by most candidates, showing that they had remembered Part I mechanics. A common mistake was to leave *k* in the answer to (b), when the question asked for the answer in terms of the other variables.
(c) This was answered well by many candidates. A common mistake was not to divide by 8π, which indicates that the candidates did not realise that the mean square velocity is the integral of S_{uu}.
(d) Most candidates had the correct method but many got the wrong answer.
(e) Most candidates showed that they understood the principle, drawing the diagram of S_{uu}, S_{ff} and S_{xx} in the notes. Most managed to draw the correct shape of $S_{xx}(\omega)$ but few put numbers onto the axes, which was the main point of the question. Most candidates correctly answered the final part of the question.