IIB 4A12 Turbulence 2010 Crib

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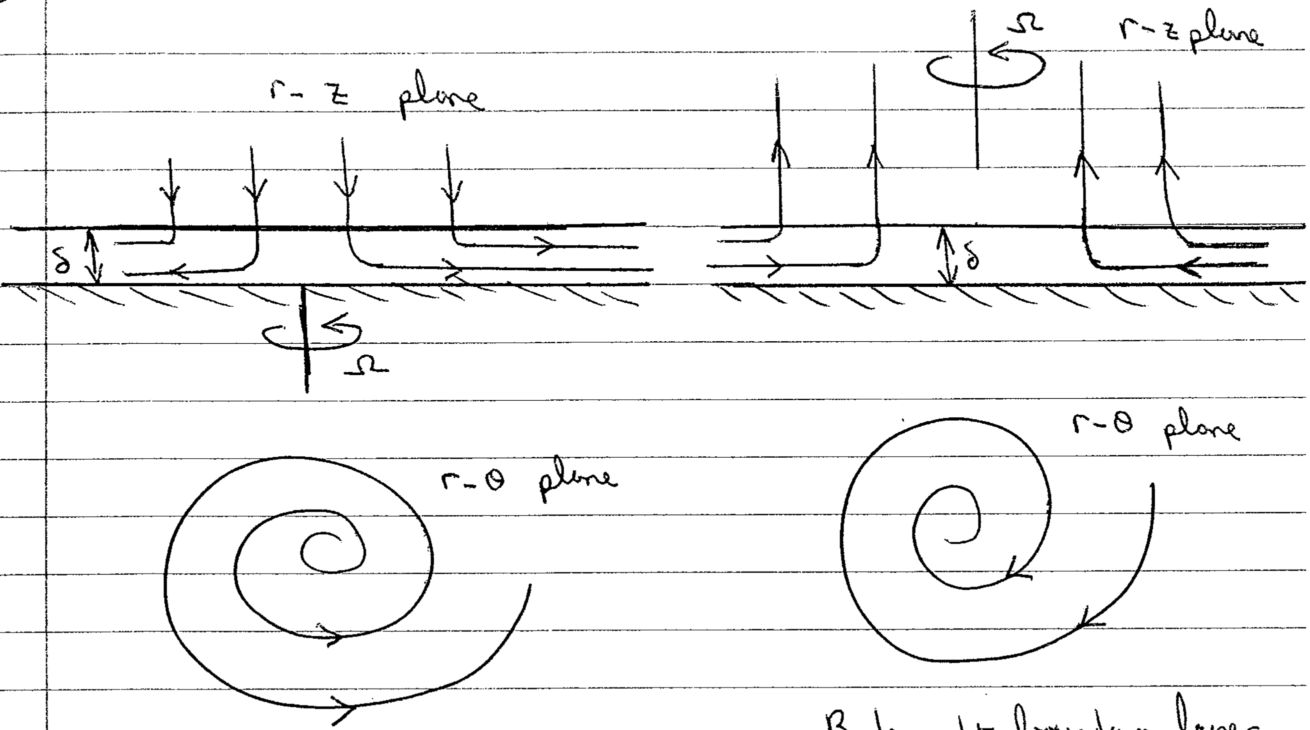
① (a)



Karman boundary layer

Bodewadt boundary layer.

Outside the boundary layer there's a balance between $\partial p/\partial r$ and $\rho u_\theta^2/r$. This sets up a radial pressure gradient which is imposed on the boundary. In the case of a Bodewadt layer there's a low pressure at $r=0$ and high pressure at large r. For a Karman layer $\partial p/\partial r = 0$. In the boundary layer this balance is disturbed because u_θ varies with z. In the Karman layer the fluid is centrifuged outward and in the Bodewadt layer u_θ is diminished so $\partial p/\partial r$ pushes the fluid inward.

[30%]

(b) In Karman layer $\partial p/\partial r = 0$ and the balance is between $\rho u_\theta^2/r$ and the viscous stress $\rho\nu\,\partial^2 u_r/\partial z^2$.

Thus,

$$\rho\frac{u_\theta^2}{r} \sim \rho\nu\frac{u_r}{\delta^2}$$

In the Bodewadt layer $\partial p/\partial r \sim \rho u_\theta^2/r$ in core, and the pressure gradient $\partial p/\partial r$ in the boundary layer is balanced by the viscous stress $\rho\nu\,\partial^2 u_r/\partial z^2$:

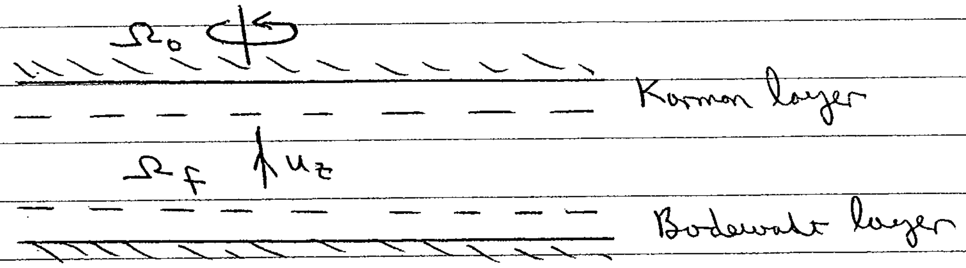
$$\frac{\partial p}{\partial r} \sim \left(\rho\frac{u_\theta^2}{r}\right)_{core} \sim \rho\nu\frac{u_r}{\delta^2}$$

In both cases $u_\theta \sim \Omega r$ and $u_\theta \sim u_r$, so,

$$\rho\frac{(\Omega r)^2}{r} \sim \rho\nu\frac{\Omega r}{\delta^2}$$

$$\Rightarrow \quad \underline{\underline{\delta \sim \sqrt{\nu/\Omega}}} \qquad [35\%]$$

(c)



$$u_z = 1\cdot4\sqrt{\nu\Omega_f} = 0\cdot9\sqrt{\nu(\Omega_0 - \Omega_f)}$$

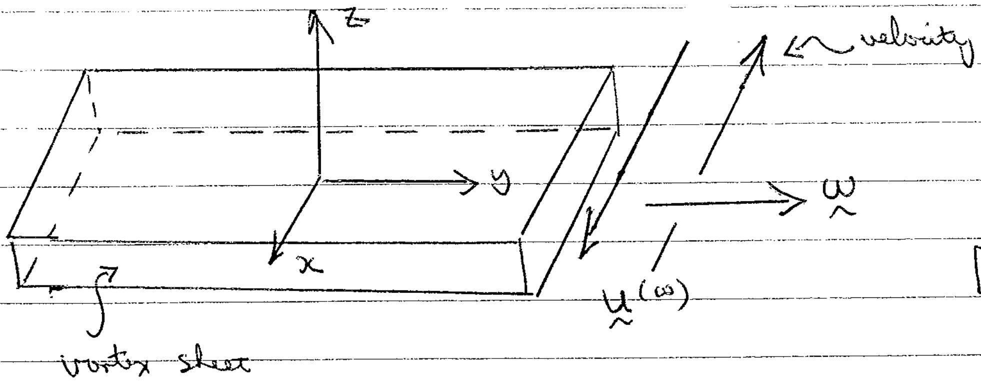
$$\Rightarrow \quad \left(\frac{1\cdot4}{0\cdot9}\right)^2 \Omega_f = \Omega_0 - \Omega_f$$

$$\Rightarrow \quad \underline{\underline{\Omega_f = 0\cdot292\,\Omega_0}} \qquad [35\%]$$

Examiner's comment:

The answers were very good by and large, with the occasional hopeless attempt. Most marks were lost in the order-of-magnitude part of the question. On reflection the question needed a sting in the tail.

② (a)



[10%]

(b) $\omega = \begin{vmatrix} i & j & k \\ 0 & 0 & \frac{\partial}{\partial z} \\ u_x & 0 & 0 \end{vmatrix} = \frac{\partial u_x}{\partial z}\hat{e}_y \Rightarrow \omega_y = \frac{\partial u_x}{\partial z}$

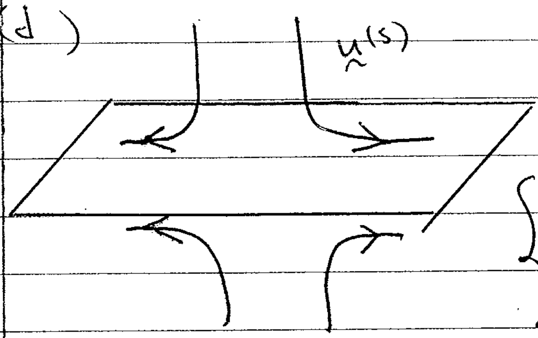
$$\begin{cases} \text{Flux} = \int_{-\infty}^{\infty} \omega_y \, dz = \int_{-\infty}^{\infty} \frac{\partial u_x}{\partial z} dz = \Delta u_x \quad \text{(velocity jump)} \\ \text{Flux} = \int_{-\infty}^{\infty} \omega_y \, dz = \frac{\Phi}{\sqrt{\pi}\,\delta} \int_{-\infty}^{\infty} e^{-z^2/\delta^2} dz = \frac{\Phi}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} ds = \Phi \end{cases}$$

[20%]

(c) $(\omega \cdot \nabla)\, u^{(\omega)} = \omega_y \frac{\partial}{\partial y} u^{(\omega)} = 0$

$(u^{(\omega)} \cdot \nabla)\, \omega = u^{(\omega)}_x \frac{\partial}{\partial x} \omega = 0$ [10%]

(d)



$u^{(s)} = (0, \alpha y, -\alpha z)$

$$\begin{cases} u^{(s)} \cdot \nabla \omega = -\alpha z \frac{\partial}{\partial z} \omega_y \hat{e}_y \\ \omega \cdot \nabla u^{(s)} = \omega_y \frac{\partial}{\partial y}(\alpha y \hat{e}_y) \end{cases}$$

$$u^{(s)} \cdot \nabla \omega = \omega \cdot \nabla u^{(s)} + \nu \nabla^2 \omega$$

$$\Rightarrow \quad -\alpha z \frac{\partial \omega_y}{\partial z} = \omega_y \alpha + \nu \frac{\partial^2 \omega_y}{\partial z^2}$$

$$\Rightarrow \quad -\alpha z\left[-\frac{2z}{\delta^2}\right]\omega_y = \alpha\omega_y + \nu\frac{\partial}{\partial z}\left[-\frac{2z}{\delta^2}\omega_y\right]$$

$$= \alpha\omega_y + \nu\left[-\frac{2}{\delta^2}\omega_y + \frac{4z^3}{\delta^4}\omega_y\right]$$

$$\Rightarrow \quad \alpha\omega_y\left[\frac{2z^3}{\delta^2} - 1\right] = \frac{2\nu}{\delta^2}\omega_y\left[\frac{2z^2}{\delta^2} - 1\right]$$

$$\Rightarrow \quad \alpha = \frac{2\nu}{\delta^2}$$

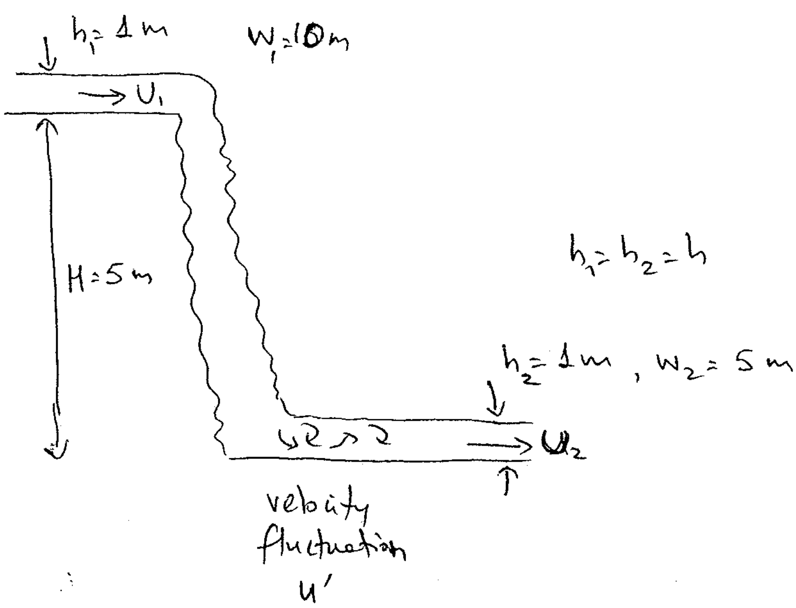
$$\Rightarrow \quad \underline{\underline{\delta = \sqrt{2\nu/\alpha}}} \qquad [40\%]$$

(e) The inward advection of vorticity counters the tendency for vorticity to diffuse outward. The stretching of vorticity maintains the vorticity at constant strength despite diffusion [20%]

Examiner's comment:

The answers were mostly sound, though curiously most marks were lost in part (b), which should have been easy.

3



(a) Continuity : $U_1 h W_1 = U_2 h W_2 \Rightarrow \underline{\underline{U_2 = 2\ m/s}}$

IMPORTANT!!

Total energy: $\frac{1}{2}\rho U_1^2 + \rho g H = \frac{1}{2}\rho U_2^2 + \frac{3}{2}\rho u'^2$

$$\Leftrightarrow 3u'^2 = 2gH - (U_2^2 - U_1^2) \Rightarrow \underline{\underline{u' = 5.63\ m/s}}$$

Turbulence "picks up" the energy from the fall (potential energy) that the mean flow didn't pick up. Note that h_2 to be equal to the one before the waterfall is probably an oversimplification of reality. (30%)

(b) A reasonable integral lengthscale is h (or a fraction of h). W_2 is not a good estimate, since it is the smallest dimension of the flow that usually serves as the order of magnitude of the turbulent lengthscale.

So : $Re_t = \frac{u' h}{\nu} = 5.6 \times 10^6$ ($\nu_{water} = 10^{-6}\ m^2/s$ at 20°C)

$\varepsilon = u'^3/h = 178.5\ m^2/s^3$

$\eta_K = h \cdot Re_t^{-3/4} = 9\ \mu m$

(40%)

(c) At the foot of the waterfall, $u' \sim 5.6$ m/s and the mean velocity is $U_2 = 2$ m/s.
Taylor's assumption states that $\frac{\partial}{\partial t} = U \frac{\partial}{\partial x}$

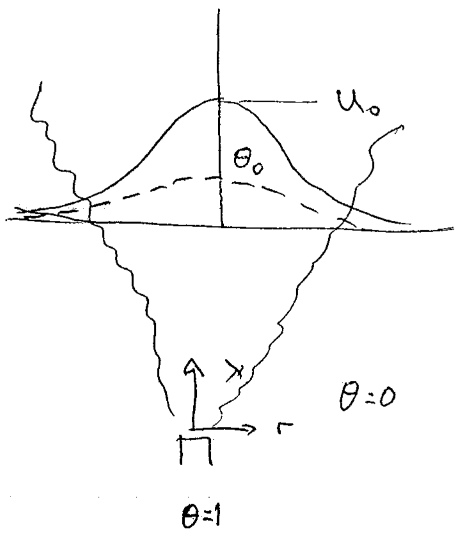
or that the turbulence is "frozen" as it is convected past an observer. It is valid if $u' \ll U_2$ which is not satisfied in our case.

Upstream, there is no turbulence so Taylor's hypothesis is OK (albeit trivial). Downstream, the turbulence will eventually decay and will become $\ll U_2$. If the flow in the river becomes something like a flat plate boundary layer, Taylor's hypothesis will be reasonable far from the wall, not very good close to the wall.

(30%)

4

(a)



$$U(x,r) = U_o(x) \cdot G(\eta)$$
$$\eta = r/\delta$$
δ = unique function of x
= characteristic width of jet

All Reynolds stresses, fluxes etc can be written as
$$F_o(x) \cdot F(\eta)$$

For the mean scalar, $\overline{\theta}(x,r) = \Theta_o(x) \cdot G_\theta(\eta)$
For the turbulence, $\overline{u'^2}(x,r) = u_o^2(x)\ G_{uu}(\eta)$
For the scalar variance, $\overline{\vartheta^2}(x,r) = \vartheta_o^2(x)\ G_{\theta\theta}(\eta)$

Self-preservation implies also:
(i) ϑ_o^2/Θ_o^2, u_o^2/U_o^2 to be constant (independent of x)
(ii) turbulence structure does not change (e.g. balance of terms in k or Re-stress transport equation stays the same)

(50%)

(b) For the round jet $\delta \sim x$.
From conservation of momentum, M = const
$$M = \int_0^\infty 2\pi r\, U^2(r)\, dr = \int_0^\infty 2\pi \eta \delta U_o^2(x)\ G^2(\eta)\ \delta\, d\eta$$
$$= \delta^2 U_o^2 \underbrace{\int_0^\infty 2\pi\eta\, G^2(\eta)\, d\eta}_{\text{Indep. of } x} = \text{const}$$

$$\Rightarrow \delta^2 U_o^2 = \text{ind of } x \Rightarrow U_o \sim x^{-1}$$
$$\Rightarrow u_o \sim x^{-1} \quad \text{since} \quad u_o/U_o = \text{const}$$

From conservation of scalar:

$$\int_0^\infty 2\pi r\, U(r)\, \Theta(r)\, dr = \text{const}$$

$$\Leftrightarrow \int_0^\infty 2\pi \eta \delta\; U_0(x)\, G(\eta)\; \Theta_0(x)\; G_\theta(\eta)\; \delta\, d\eta = \text{const}$$

$\Leftrightarrow$ $\delta^2 U_0 \Theta_0$ must be independent of x

$\Rightarrow \Theta_0 \sim x^{-1}$

$\Rightarrow \vartheta_0 \sim x^{-1}$

Scalar dissipation is modelled as

$$N = C \cdot \frac{g^2}{T_{turb}}, \quad T_{turb} = \frac{L_{turb}}{u'}$$

$$= \frac{\delta}{u_0} \quad \text{for this problem}$$

$$\Rightarrow N \sim \frac{\vartheta_0^2 u_0}{\delta} \sim \underline{\underline{x^{-4}}}$$

Scalar dissipation decays very quickly downstream.

(50%)