Engineering Tripos Part IIB 2010
Module 4C6 Advanced Linear Vibration
Solutions
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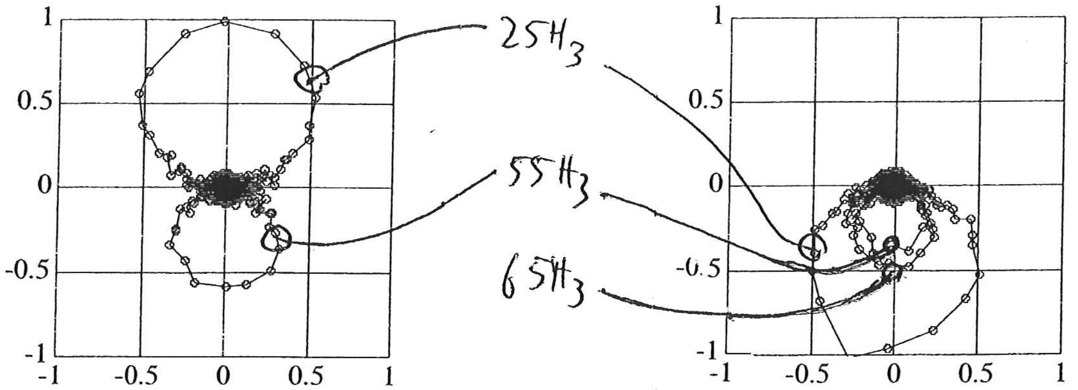
1(a) Identify circles by comparing the maximum amplitude on the circle with the peak height in the magnitude plots:



(b) (i) Sampling rate is 320 Hz, 8192 data points, so total time window 25.6 s, and frequency resolution 1/25.6 = 0.039 Hz. This gives the spacing of the individual frequency points on the modal circles. To estimate Q factors, it is not possible to see bandwidths from the magnitude plots, but it is easy to count points around the top half of each circle. That converts to a half-power bandwidth *B*, then $Q = F/B$ where *F* is the centre frequency. The results are as follows:

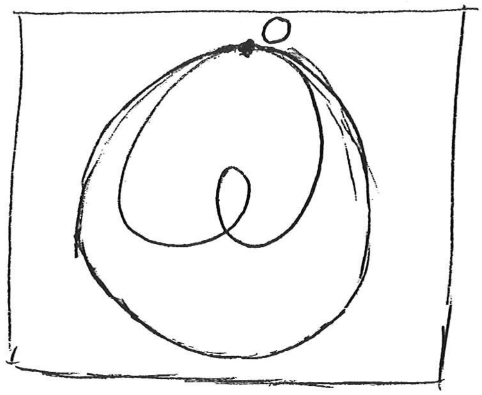
	Grid point	1	1	1	2	2	2
Mode	Freq (Hz)	Number	*B* (Hz)	*Q*	Number	*B*	*Q*
1	25	6.5	0.25	99	6	0.234	107
2	55	6	.234	235	6	.234	235
3	65	-	-	-	8	.312	208

Given the uncertainty of the estimation method, one would not want to claim more than that mode 1 has a Q around 100, and modes 2 and 3 have Qs around 200.

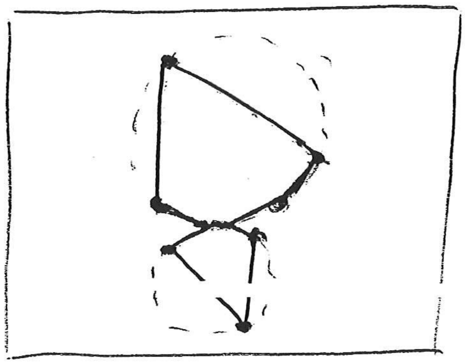
(ii) For small damping, as seen here, the peak height (or diameter of each circle) can be deduced from the Data Sheet expression on P2 section 7 by substituting $\omega = \omega_n$ to give the peak amplitude at resonance. This is an acceleration measurement, so an extra factor $-\omega^2$ is needed, multiplying the entire expression. The result is that the value at resonance is approximately $iu_n(x)u_n(y)/(2\varsigma_n) = iu_n(x)u_n(y)Q_n$. The sign of the modal factor must be deduced from the orientation of the circle (positive for upwards, negative for downwards). The modal factors can thus be deduced:

	Grid point	1	1	2	2
Mode	Freq (Hz)	Diameter	Factor	Diameter	Factor
1	25	1	0.01	0.9	–0.009
2	55	0.6	–0.003	0.35	–0.0018
3	65	-	-	0.45	–0.0023

(c) (i) A frequency difference of 0.5 Hz is somewhat wider apart than the half-power bandwidth of each peak, so modest overlap occurs. The two circles are superimposed and consequently distorted. If the modal amplitudes stay the same, then grid point 1 is unchanged because the 65 Hz mode does not show there. Grid point 2 will have something like:



(ii) If the sampling rate is increased without changing the total number of samples, then the spacing of points around the circles will decrease. A factor 10 in the sampling rate means that we only see every tenth point, giving rather "sketchy" circles: e.g. for grid 1:



(d) Amplitude range from highest peak to noise floor is about a factor of 10, so 20 dB. This is much less good than in the lab measurements for this module, and it should be possible to improve. Noise level is influenced by: mechanical background noise in the system; electrical noise pickup, from cables and also within the data-logger computer; choice of gain factor in the hammer and accelerometer, combined with the electrical signal-to-noise ratio for the amplifiers and any filters used. In a nuclear power plant, if it is running, there may be mechanical noise from circulating coolant etc. The way to reduce the effect of that is by averaging a large number of takes of each transfer function. The signature of the noise in this case does not look very much like electrical pickup: no obvious peaks at multiples of 50 Hz (or 60 Hz if the reactor is on a ship or submarine). If this were a problem, the cures involve careful shielding of cables, attention to earthing arrangements, and perhaps running the data-logger on battery power rather than mains power to reduce internal pickup. Choosing the right accelerometer and hammer force sensor is a matter of monitoring the voltages generated with typical impulses. The signal needs to be such that the peak levels are not in danger of clipping, but then should be as large as possible within that constraint to minimise the influence of the amplifier and sensor noise floors.

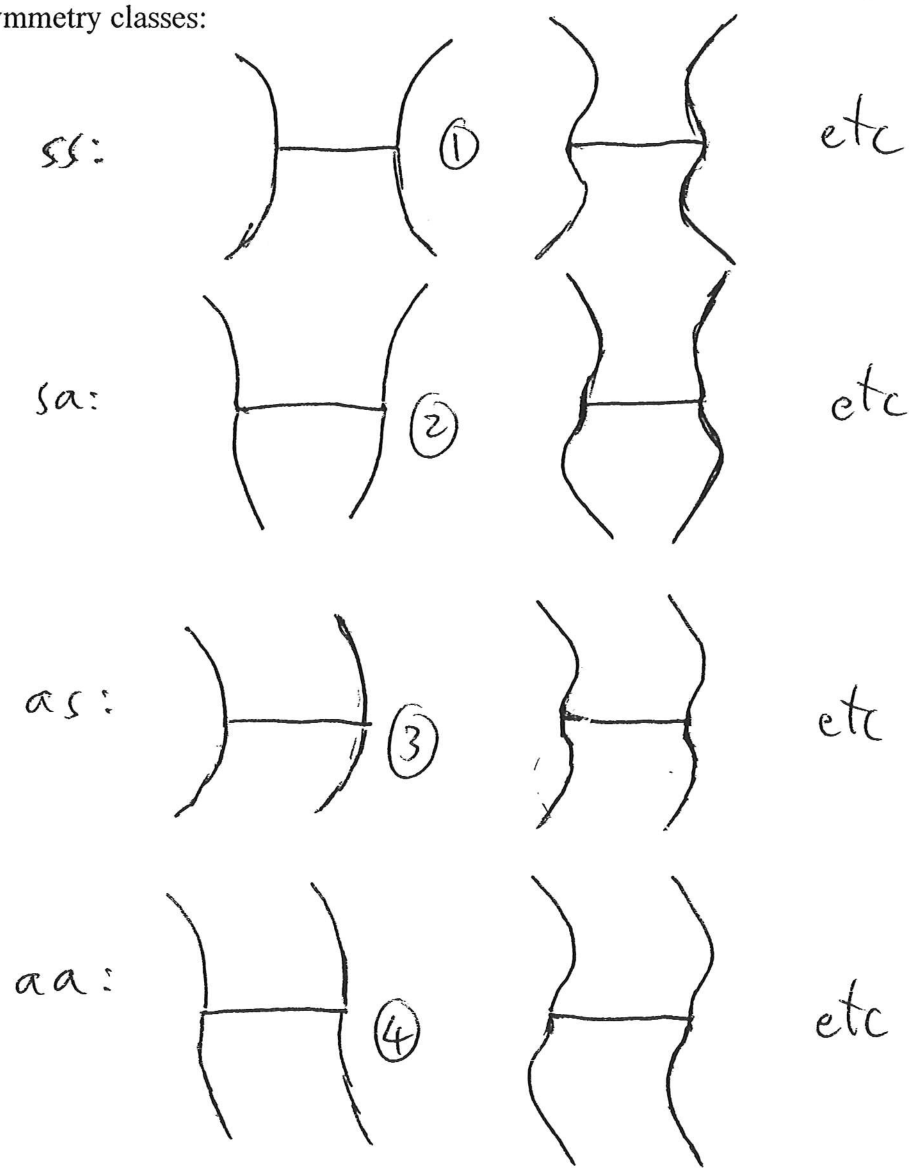
2 (a) Many examples are possible, some detailed in the lecture notes. For example:

(I) Constrain one point on a system to be motionless. Number of degrees of freedom is reduced by 1, and new frequencies interlace the old ones.

(II) Introduce a coupling link between two systems as in the 3C6 lab experiment. New (coupled) frequencies will interlace the combined list of frequencies from the original two systems.

(III) Change one boundary condition on a bending beam in a way which adds or relaxes one constraint: e.g. pinned to clamped (add one constraint) or pinned to free (remove one constraint). Provided only a single change is made (i.e. on end but not both) then the new frequencies will interlace the old ones. If two constraints are added (e.g. free to clamped) then interlacing occurs at each separate step, but the final results need not strictly interlace the original ones.

(b) (i) The system has two planes of mirror symmetry. It must be possible to represent all vibration modes in a way which is either symmetric or antisymmetric in EACH of the planes separately. So there are four classes of mode: sym/sym, sym/anti, anti/sym, anti/anti. For this system, we expect the modes (nearly all) to be based around the individual resonances of the separate beams AC, AD etc. There are four identical beams, so there should be 4 modes associated with each individual resonance. Example sketches in the four symmetry classes:



(ii) If the link AB is held rigidly, the four separate bending beams are entirely decoupled from each other, so the frequencies must be the same as those of the individual beams. So there are 4 modes at each frequency. Because they are equal, any linear combination of the mode shapes will also be a mode at that same frequency. So the modes could be represented as vibration of each cantilever beam separately, but they can equally well be represented in a way which preserves the symmetries discussed in (i) above.
(iii) Making the spring finite relaxes one constraint (or equivalently, starting from the finite spring we would need to impose one constraint to make it rigid, by preventing lateral motion of AB). So the new frequencies must interlace. But they were in groups of 4, so there must still be 3 modes at each of the original frequencies, with the 4th mode of the group being shifted downwards into the gap between its original frequency and the next lower frequency. Looking at the modes from part (i), the ones numbered 1,2,4 are unchanged by making the spring finite because they exert no net force on the spring, but the mode 3 will now involve some motion of the spring and will shift to a lower frequency.

3 (a) In a bolted structure like Meccano, with individual elements made of material with relatively low damping, the main source of dissipation is likely to come from the joints: micro-friction and perhaps air-pumping at the bolted lap joints. Additional damping will come from the material: the steel has relatively low intrinsic damping, but Meccano usually has a paint finish of some kind on the elements, and this might contribute higher material damping depending on the material of the paint. If the structure is relatively light and flimsy, additional damping may come from air viscosity.

(i) To add damping: leave the bolts loose to increase frictional effects or induce rattling; add high-loss rubber washers at the bolted joints; coat the whole structure in something lossy like bitumen.

(ii)To reduce damping is hard, but it would depend on details around the joints. Gluing instead of bolting would help, provided the glue was not highly lossy: something brittle like Superglue would help. Possibly adding "spiky" washers at the joints would help, by keying the surfaces together firmly and reducing micro-slipping, and also by spacing the components wider apart to reduce air pumping effects.

(b) MEMS devices are fabricated in a monolithic way, by etching the moving shape out of the solid material (silicon, usually). So there are no joints to contribute boundary damping, in contrast to the Meccano. The intrinsic damping of the material is low, and material damping associated with that puts a lower limit on modal damping. But various effects may in practice increase the damping: there may be micro-cracks, there may be deposited layers of different material (e.g. gold for contact electrodes) which might have higher damping than silicon, there may be energy loss from the intended resonant device because of vibration transmission into the rest of the chip, and if the device is not operating in high vacuum there will be viscous dissipation in the surrounding gas.

(i) It is unusual to want to add damping to a MEMS device, but it could be done by deliberately enhancing the effects just described. Deliberate micro-cracks are probably undesirable because of reducing the fatigue life, but some kind of deposited layer (making a free or constrained layer damper) could added. The vacuum level could be reduced (e.g. by packaging in dry Nitrogen). The support structures could be designed in a relatively floppy way to encourage vibration transmission.

(ii) Minimising damping is the usual requirement in these devices. Very careful detailing is needed to avoid vibration loss into the attachment points of the resonant device. This is especially true if the resonance is based on axial rather than bending motion. Supporting near nodal points (as in a tuning fork) is often done. Tethering springs are often "convoluted" shapes to make them as floppy as possible and minimise coupling to surrounding structures. The quality of the silicon substrate, and stages of the fabrication process, may influence the presence of microcracks or other imperfections.

(c) Railway carriages have suspension systems with deliberate damping incorporated, as with other vehicles. There are two main types: piston-like dampers similar to the "shock absorbers" seen on road vehicles, and frictional dampers which convert vertical motion in the suspension springs into rotational motion between "clutch plates". Some additional damping will be contributed by the other moving parts of the suspension system: attachment bushes for the springs, friction in sliding or rotating bearings. Other damping sources are negligible by comparison.

(i), (ii) Because the damping is provided mainly by a designed component, separate from the rest of the system, the damping level can be increased or decreased by adjusting the detailed design of this element in fairly obvious ways. For example the viscous dissipation in piston-type dampers is determined by the internal fluid and the flow details induced by piston motion. For example, viscous effects can be increased or decreased by adjusting orifice sizes and geometry.

4(a) Governing equation from Data Sheet:

$$T\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) = m\frac{\partial^2 w}{\partial t^2}$$

So try $w = X(x)Y(y)e^{i\omega t}$

$$\therefore T\left(X''Y + XY''\right) = -m\omega^2 XY$$

$$\therefore \frac{X''}{X}+\frac{m\omega^2}{T} = -\frac{Y''}{Y} = \text{constant} = k_2^2 \text{ say}$$

(choose positive constant to get trig behaviour in y)

Then $Y'' = -k_2^2 Y$ so $Y = \left.\begin{matrix}\sin\\ \cos\end{matrix}\right) k_2 y$

$Y(0) = Y(b) = 0$, so want $\sin k_2 y$, and need $\sin k_2 b = 0$

so possible values are $k_2 = n\pi/b \quad n = 1, 2, 3 \ldots$

Then $\frac{X''}{X} = -m\omega^2 + \frac{n^2\pi^2}{b^2} = -k_1^2$ say ①

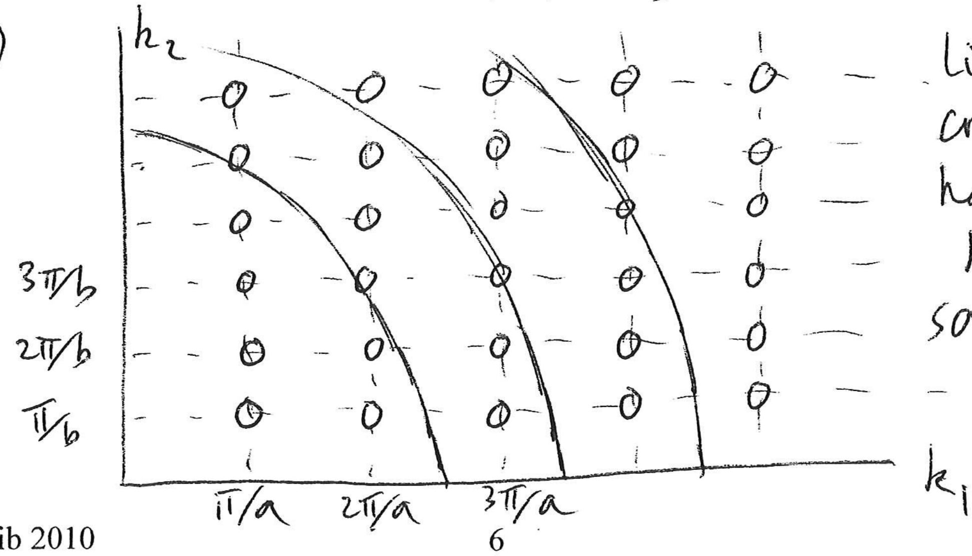
Then $X = \sin k_1 x$ by same argument, with $\sin k_1 a = 0$

so $k_1 = \frac{m\pi}{a}$, $m = 1, 2, 3 \ldots$

Then require $\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2} = \frac{m\omega^2}{T}$ from ①

determines the natural frequency : ie $m\omega^2 = k_1^2 + k_2^2$

(b)



Lines of constant ω have constant $k_1^2 + k_2^2$, so circles.

4 cont. So the allowed values cover the (k_1, k_2) plane in a regular grid, one point per area $\frac{\pi}{a} \times \frac{\pi}{b}$, so density of points is $\frac{ab}{\pi^2}$ per unit area.

So number of modes $< \omega$ is roughly given by this density × area of ¼-circle at ω:

ie (number $< \omega$) $\approx \frac{1}{4}\pi\left(\frac{m\omega^2}{T}\right) \times \frac{ab}{\pi^2} = \frac{ab\,m\omega^2}{4\pi T}$

Call this function $N(\omega)$.

Now number of modes between ω and $\omega + \delta\omega$ is approximately $\frac{dN}{d\omega}\delta\omega$, so the density of modes per unit interval of ω is $\frac{dN}{d\omega} = \frac{ab\,m\omega}{2\pi T}$

4C6 2010: Answers

Q1: (b)(i) Q factors approximately 100, 200, 200.
(b)(ii) Factors for grid point 1 0.01, -0.003, -; for grid point 2 -0.009, -0.0018, -0.0023

Q4 (c) Number of modes below ω is approximately $\dfrac{abm\omega^2}{4\pi T}$

Modal density approximately $\dfrac{abm\omega}{2\pi T}$

Assessor's comments on questions, module 4C6 2010

Q1 Experimental methods and modal analysis

Attempted by virtually all candidates. Generally well done, but remarkably few candidates mentioned *averaging* as a way to reduce measurement noise, despite most presumably having done the experiment in IIA 3C6 including this topic.

Q2 Interlacing theorem

Few candidates took advantage of the fact that the structure has two symmetry planes, so that modes come in groups of 4 with the various combinations of symmetric/antisymmetric. Nevertheless many produced generally plausible sketches of the low mode shapes. Many were not really clear how to apply the interlacing theorem to this problem.

Q3 Mechanisms of vibration damping

Reasonably good efforts on the qualitative discussions. Interesting variations of thinking on the three application problems. Surprisingly many had not noticed that trains generally have suspension systems.

Q4 Membrane vibration and modal density

Good efforts on the bookwork. A lot of good attempts on the k-plane sketch, but relatively few grasped the modal density argument clearly.