PART IIB 2010 4C7 RANDOM AND NON-LINEAR VIBRATIONS

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1 a) $M\dot{v} + Cv = F$

Put $v(t) = v(i\omega)e^{i\omega t}$ and $F(t) = F(i\omega)e^{i\omega t}$

$\Rightarrow \quad Mi\omega v + Cv = F \quad \Rightarrow \quad \underline{H(i\omega) = \frac{v(i\omega)}{F(i\omega)} = (Mi\omega + C)^{-1}}$

Now $S_{vv}(\omega) = |H(i\omega)|^2 S_{FF}(\omega)$

$\Rightarrow \quad \underline{S_{vv}(\omega) = \frac{S_{FF}(\omega)}{M^2\omega^2 + C^2}}$

For rms velocity : $\sigma_v^2 = \int_0^\infty S_{vv}(\omega)d\omega = \underbrace{\int_{\omega_1}^{\omega_2} \left(\frac{S_0}{M^2\omega^2 + C^2}\right) d\omega}_{\text{Put } M\omega/C = x \;\Rightarrow\; dx = (M/C)\,d\omega}$

$\sigma_v^2 = S_0\left(\frac{1}{CM}\right) \int_{M\omega_1/C}^{M\omega_2/C} (1+x^2)^{-1} dx$

$\underline{\sigma_v^2 = \left(\frac{S_0}{CM}\right)\left[\tan^{-1}(M\omega_2/C) - \tan^{-1}(M\omega_1/C)\right]}$ [40%]

b) $\dot{v}(i\omega) = i\omega v(i\omega) \quad \Rightarrow \quad \frac{\dot{v}(i\omega)}{F(i\omega)} = \frac{i\omega}{Mi\omega + C}$

$\Rightarrow \quad S_{\dot{v}\dot{v}}(\omega) = \frac{\omega^2 S_{FF}(\omega)}{M\omega^2 + C^2} \quad \Rightarrow \quad \sigma_{\dot{v}}^2 = \underbrace{\int_{\omega_1}^{\omega_2} \frac{\omega^2 S_0}{M^2\omega^2 + C^2} d\omega}_{\text{Again, put } x = M\omega/C}$

$\sigma_{\dot{v}}^2 = \frac{S_0 C}{M^3} \int_{M\omega_1/C}^{M\omega_2/C} \underbrace{\frac{x^2}{1+x^2}}_{\text{integrand is } 1 - \frac{1}{1+x^2}} dx$

$\Rightarrow \quad \underline{\sigma_{\dot{v}}^2 = \frac{S_0 C}{M^3}\left[(M/C)(\omega_2 - \omega_1) - \tan^{-1}(M\omega_2/C) + \tan^{-1}(M\omega_1/C)\right]}$ [30%]

c) Substituting numerical data $\Rightarrow \quad \sigma_v^2 = 0.798 \qquad \underline{\sigma_v = 0.894 \text{ m/s}}$

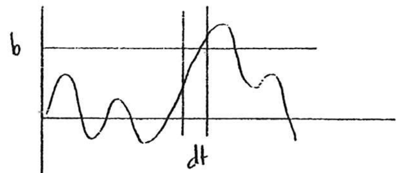
and $\sigma_{\dot{v}}^2 = 0.1598 \qquad \underline{\sigma_{\dot{v}} = 0.399 \text{ m/s}^2}$ [20%]

d) Mean rate of crossing zero with positive slope $= \left(\frac{1}{2\pi}\right)\left(\frac{\sigma_{\dot{v}}}{\sigma_v}\right) = 0.0723 = 1/T_{av}$

$\Rightarrow \quad \underline{T_{av} = 13.83 \text{ s}}$ [10%]

2 a) Initially, calculate the probability that an impact will occur in a time interval $dt$:

Conditions for an impact with velocity greater than $V$ are: (i) $\dot{x} > V$
(ii) $b - \dot{x}\,dt \le x \le b$




Assuming $\dot{x} \simeq$ constant over $dt$, $x$ must lie between $b - \dot{x}\,dt$ and $b$ at the time $t$.

Probability of meeting conditions $= \int_V^\infty \int_{b-\dot{x}dt}^{b} p(x,\dot{x})\,dx\,d\dot{x} \xrightarrow{\text{small } dt} \left\{ \int_V^\infty \dot{x}\, p(b,\dot{x})\,d\dot{x} \right\} dt$

Now, probability of impact in $dt$ = Mean rate of impacts $\times\ dt = \nu dt$

$$\Rightarrow \quad \underline{\nu = \int_V^\infty \dot{x}\, p(b,\dot{x})\,d\dot{x}}$$

[30%]

b) $x(t)$ Gaussian $\Rightarrow$ $p(x,\dot{x}) = \frac{1}{2\pi\sigma_x\sigma_{\dot{x}}} e^{-\frac{1}{2}(x/\sigma_x)^2} e^{-\frac{1}{2}(\dot{x}/\sigma_{\dot{x}})^2}$

$$\Rightarrow \nu = \frac{1}{2\pi\sigma_x\sigma_{\dot{x}}} e^{-\frac{1}{2}(b/\sigma_x)^2} \underbrace{\int_V^\infty \dot{x}\, e^{-\frac{1}{2}(\dot{x}/\sigma_{\dot{x}})^2}\,d\dot{x}}_{\left[-\sigma_{\dot{x}}^2 e^{-\frac{1}{2}(V/\sigma_{\dot{x}})^2}\right]_V^\infty}$$

$$\Rightarrow \underline{\nu = \frac{1}{2\pi}\left(\frac{\sigma_{\dot{x}}}{\sigma_x}\right) e^{-\frac{1}{2}(b/\sigma_x)^2} e^{-\frac{1}{2}(V/\sigma_{\dot{x}})^2}}$$

[30%]

c) Standard results for response to white noise: $\sigma_x^2 = \frac{\pi S_0}{M^2 4\beta\omega_n^3} = \frac{\pi S_0}{2M^2(C/M)(k/M)} = \underline{\frac{\pi S_0}{2Ck}}$

$\sigma_{\dot{x}}^2 = \omega_n^2\,\sigma_x^2 = \frac{\pi S_0}{2Ck}\left(\frac{k}{M}\right) = \underline{\frac{\pi S_0}{2CM}}$

For given data $\sigma_x^2 = 1.16\times10^{-6} \Rightarrow \underline{\sigma_x = 1.078\times10^{-3}\ \text{m}}$

$\sigma_{\dot{x}}^2 = 2.618 \Rightarrow \underline{\sigma_{\dot{x}} = 1.618\ \text{m/s}}$

$\nu = \frac{1}{2\pi}\left(\frac{1.618}{1.078\times10^{-3}}\right) e^{-\frac{1}{2}(5/1.078)^2} e^{-\frac{1}{2}(5/1.618)^2} = \underline{2.066\times10^{-3}}$

[40%]

$P_{impact} = 1 - e^{-\nu t} = 1 - e^{-\nu\times10} = \underline{0.0206}$ ; Reccalculate with $b = 3$mm gives $\underline{P_{imp} = 0.3}$

Q3

(a)

$$V(x) = \frac{1}{2}\left[2\sqrt{(L/2)+x^2} - L_0\right]^2$$

$F(x)$ = restoring force of spring

$$= \frac{dV}{dx}$$

$$= -4kx\left(1 - \frac{L_0}{\sqrt{L^2+4x^2}}\right).$$

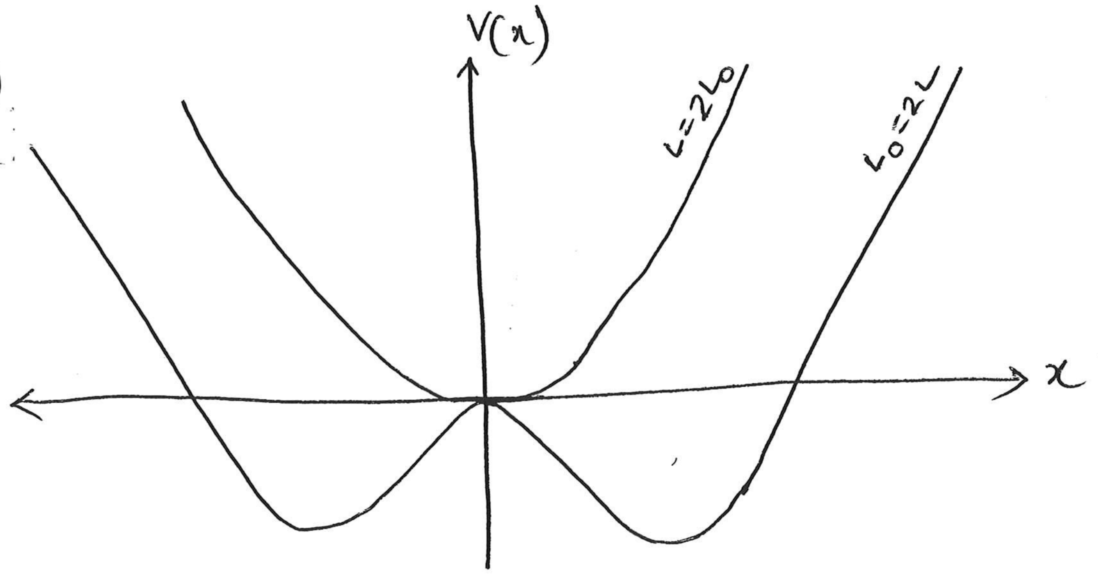
$$= -4kx\left(1 - \frac{L_0}{L\left(1+\frac{4x^2}{L^2}\right)^{1/2}}\right).$$

For small values of $x$ relative to $L$ :-

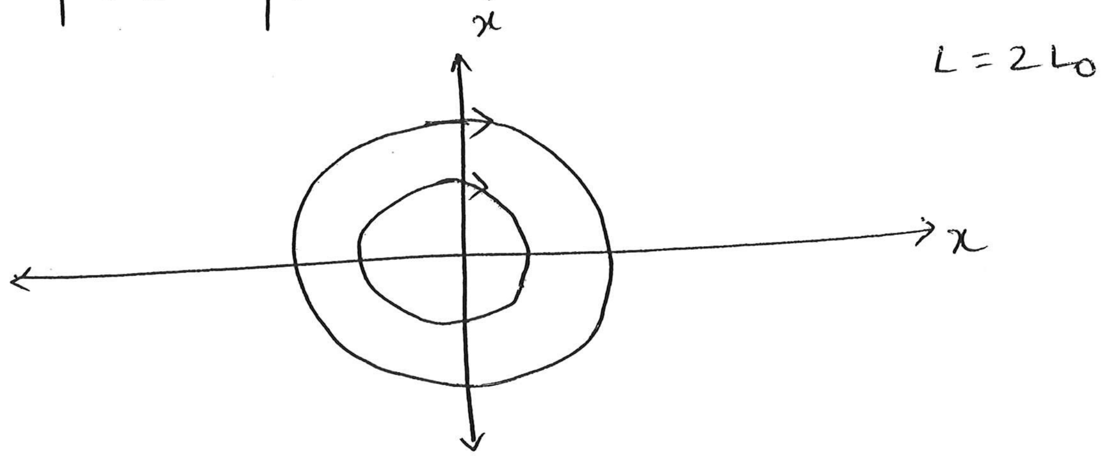
$$= \frac{-4kx(L-L_0)}{L} - 8kL_0\left[\left(\frac{x}{L}\right)^3 - 3\left(\frac{x}{L}\right)^5 + \dots\right.$$

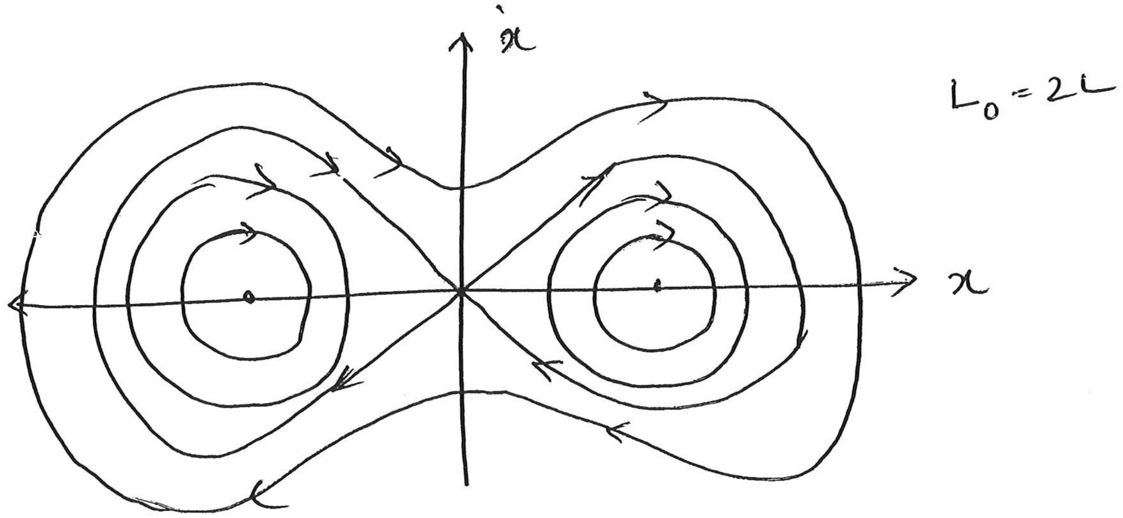
Notice that the first higher order term in the expansion is a cubic (similar to Duffing)

(b)



The phase portraits are:




For $L_0 < L$ the equilibrium point $x=0$ is stable and the potential energy curve has a minimum. For $L_0 > L$, the spring is compressed at $x = 0$ and the potential energy curve has a maximum. For $L_0 = 2L$, two stable equilibrium points are obtained at approximately $x = \pm L$ about which oscillation occurs for small energy input.

Q4(a) $\ddot{x} + \epsilon(3x^2-1)\dot{x} + p^2 x = 0$.

zero order (set $\epsilon = 0$) $\therefore x = A\cos pt$.

First order: $\ddot{x} + p^2 x = -\epsilon(3A^2\cos^2 pt - 1)\sin pt$

$$= \frac{3A^3 \mu p^2}{4} \sin 3pt - A\mu p^2\left(1 - \frac{3A}{4}\right) \times \sin pt$$

Substituting $x = \alpha \sin 3pt + \beta t \cos pt$

$$\dot{x} = 3p\alpha \cos 3pt + \beta \cos t - \beta t p \sin pt$$

$$\ddot{x} = -9p^2\alpha \sin 3pt - 2\beta p \sin pt - \beta p^2 t \cos pt.$$

substitute and equate coefficients of $\sin 3pt$, $\sin pt$

$$-9p^2\alpha + p^2\alpha = \frac{3A^3\mu p^2}{4}$$

$$-2\beta p = A\mu p^2\left(1 - \frac{3A^2}{4}\right).$$

$\beta \neq 0 \rightarrow$ growing amplitude solution with steady solution @ $\frac{3A^2}{4} - 1 = 0$ or $A^2 = 4/3$

(b) $x = A\sin pt + B\sin(\omega t + \phi)$

Substituting into the van der Pol equation and retaining terms at frequencies $p$ and $\omega$ only:-

$$(p^2B - \omega^2 B)\sin(\omega t+\phi) + 3\epsilon\left(A^2\frac{(1-\cos 2pt)}{2} + B^2\frac{(1-\cos 2(\omega t+\phi))}{2}\right.$$
$$\left.\times\left(pA\cos pt + \omega B\cos(\omega t+\phi)\right)\right)$$
$$-\epsilon\left(pA\cos pt + \omega B\cos(\omega t+\phi)\right) = \epsilon F\cos\omega t$$

Equating the coefficients of the same frequency on LHS and RHS:

$$\frac{3}{4}(2B^2+A^2) = 1 \qquad \text{— (1)}$$

$$p^2B - \omega^2 B = -\epsilon F\sin\phi \qquad \text{— (2)}$$

$$\frac{3}{2}A^2\omega B + \frac{3\omega B^3}{4} - \omega B = F\cos\phi \qquad \text{— (3)}$$

(c) squaring and adding (2) and (3) to eliminate $\phi$.

$$\frac{1}{\epsilon^2}(p^2B - \omega^2 B)^2 + \left[\frac{3}{8}(2B^2+A^2) - 1\right]^2\omega^2B^2 = F^2 \qquad \text{— (4)}$$

It is clear that for a given value of $\omega$ there is a bound on the magnitude of $F$ for which a solution of this type is possible. If this bound is exceeded, the only possibility is $A=0 \Rightarrow$ the limit cycle is quenched.

Assessor's comments:

**Questions:**

Question 1 This question was attempted by most students. Many students had difficulty with working out the resulting integral for the r.m.s. velocity and acceleration in parts (a) and (b). Other students confused definitions of single-sided and double-sided spectral density.

Question 2, This question was very popular and attempted by all students with a high average mark. The first part of this question was generally well done. Some students were unable to solve the resulting integral in part (b); others made mistakes in calculating the probability of impact damage in part (c).

Question 3, Most students showed a good grasp of underlying concepts and physical intuition in solving this problem. Common mistakes involved an incorrect estimate of the force in part (a) and then difficulty with translating the resulting expression to a series expansion in $x$ to reduce it to the Duffing form. The phase plane diagrams were generally well done by those who attempted part (b).

Question 4, This question was the least popular and attempted by the fewest number of candidates. Many students had difficulty with the harmonic balance analysis in part (b) and a few students appeared to have run out of time in answering this question.