CONTINUUM MECHANICS

Part IIB 2010 - 4C9 - Solutions PROF J A WILLIAMS

1. (a)(i) $A_{ij} = \delta_{ij} B_{kk} + 3 B_{ij}$

$3B_{ij} = A_{ij} - \delta_{ij} B_{kk}$

$\therefore 3B_{ii} = A_{ii} - \delta_{ii} B_{kk}$

But $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$ & $B_{kk} = B_{ii}$

$\therefore 3B_{ii} = A_{ii} - 3B_{ii}$

$$B_{ii} = \frac{1}{6} A_{ii}$$

Now $A_{ij} = \delta_{ij} B_{kk} + 3B_{ij}$

$= \delta_{ij} \frac{A_{kk}}{6} + 3B_{ij}$

$$\therefore B_{ij} = \frac{1}{3} A_{ij} - \frac{1}{18} \delta_{ij} A_{kk}$$

(ii) $x_{,j} \equiv \frac{\partial x}{\partial x_1} \underline{e}_1 + \frac{\partial x}{\partial x_2} \underline{e}_2 + \frac{\partial x}{\partial x_3} \underline{e}_3$

But $x^2 = x_1^2 + x_2^2 + x_3^2$

$\therefore 2x \frac{\partial x}{\partial x_1} = 2x_1 \frac{\partial x_1}{\partial x_1}$

$\frac{\partial x}{\partial x_1} = \frac{x_1}{x}$

$\therefore x_{,j} = \frac{x_1}{x} \underline{e}_1 + \frac{x_2}{x} \underline{e}_2 + \frac{x_3}{x} \underline{e}_3$

i.e. $x_{,j} = \frac{x_j}{x}$

Q1 (b) (i)

Drucker's concept of stability

Consider an elastic-plastic solid under a stress $\overset{*}{\sigma}_{ij}$ which may be in the elastic domain or may bring the solid to the point of yield. Now apply an arbitrary additional stress $\Delta\sigma_{ij}$ due to an external agency. Then, the total stress at any instant is $\sigma_{ij} = \overset{*}{\sigma}_{ij} + \Delta\sigma_{ij}$, and the plastic strain ε_{ij}^{PL} evolves with the history of σ_{ij}. The work done by the external agency in a *closed stress cycle* of $\Delta\sigma_{ij}$ is

$$W_{ea} = \oint \Delta\sigma_{ij} d\varepsilon_{ij}$$

A closed cycle of loading for the additional stress $\Delta\sigma_{ij}$ generates a closed cycle of loading for the total stress $\sigma_{ij} = \overset{*}{\sigma}_{ij} + \Delta\sigma_{ij}$.

Drucker argued that a stable plastic solid has the property that $W_{ea} \geq 0$**.** It reduces to the idea that a stable solid strain hardens, or is elastic, ideally plastic but cannot strain soften.

Drucker's stability argument leads to two Postulates. The postulates dictate convexity and normality of the yield surface. Let the plastic strain rate be $\dot{\varepsilon}_{ij}^{PL} \geq 0$, and the stress state that is associated with this strain rate is σ_{ij}. Also let $\overset{*}{\sigma}_{ij}$ be some other stress state within or on the yield surface. Then $\left(\sigma_{ij} - \overset{*}{\sigma}_{ij}\right)\dot{\varepsilon}_{ij}^{PL} \geq 0$. Additionally, $\dot{\sigma}_{ij}\dot{\varepsilon}_{ij}^{PL} \geq 0$.

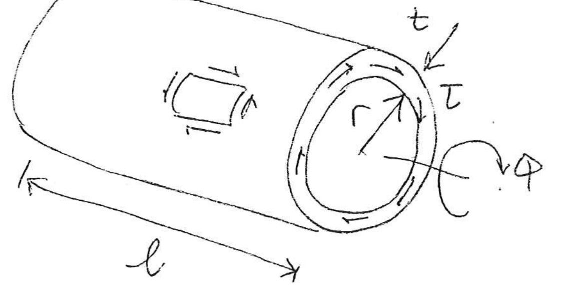
For a stable plastic solid, they must hold for all pairs $\left(\dot{\sigma}_{ij}, \dot{\varepsilon}_{ij}^{PL}\right)$ of loading, for every combination of $\left(\sigma_{ij}, \overset{*}{\sigma}_{ij}\right)$.

There are 2 consequences: the yield surface is convex and $\dot{\varepsilon}_{ij}^{PL}$ lies normal to the yield surface.

Q1(b) (ii)

Let relative angular twist per unit length be α .

Shear strain in wall of tube $\gamma = \alpha r / \ell$



$\phi = 2\pi r^2 t \cdot \tau$

In elastic regime

$\tau = G\gamma = \frac{Gr\alpha}{l}$

$\phi =$ torque
$\alpha =$ angle of twist

$\therefore \quad \phi = \frac{2\pi G \alpha r^3 t}{l}$

principle stresses $(\tau, -\tau, 0)$

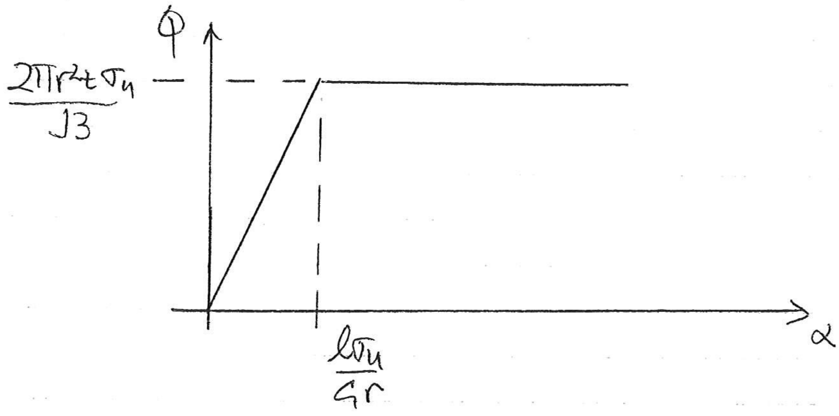
$\therefore$ V.M. $\quad [\tau - (-\tau)]^2 + [(-\tau) - 0]^2 + [0 - \tau]^2 = 2\sigma_y^2$

i.e. $6\tau^2 = 2\sigma_y^2$

$\tau = \frac{\sigma_y}{\sqrt{3}}$

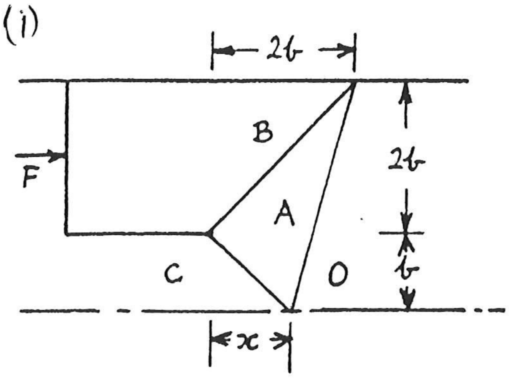
At first yield $\quad \phi = \frac{2\pi r^2 t \sigma_y}{\sqrt{3}}$ and $\alpha = \frac{l\tau}{Gr} = \frac{l\sigma_y}{\sqrt{3}Gr}$

Post first yield torque remains constant



Examiner's comments Part (a) was well done. Some explanations of Druker a bit garbled & some surprisingly weak attempts to the very easy final part.

2 (i)



$$F v_B = k\left(l_{AC} v_{AC} + l_{OA} v_{OA}\right)$$

$$F\left(b - \tfrac{1}{3}(2b - x)\right) =$$

$$k\left(\sqrt{x^2+b^2}\sqrt{x^2+b^2} + \frac{\sqrt{(2b-x)^2+9b^2}\sqrt{(2b-x)^2+9b^2}}{3}\right)$$

$$\therefore F/k = 4\left(\frac{x^2 - bx + 4b^2}{x+b}\right)$$

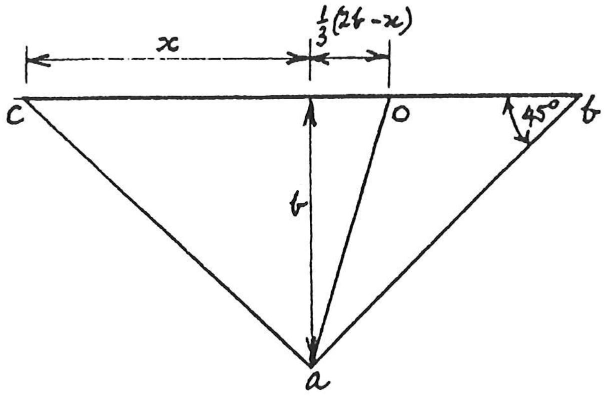
Minimum value when

$\frac{d}{dx}\left(F/k\right) = 0$ gives

$x = (\sqrt{6} - 1)b = 1.45b$

$F_{min} = 4kb\left(\frac{12 - 3\sqrt{6}}{\sqrt{6}}\right) = 7.6kb$

Total for $2F_{min} = \underline{\underline{15.2kb}}$



Choose scale of velocity diagram so that vertical velocity of point a is proportional to b.

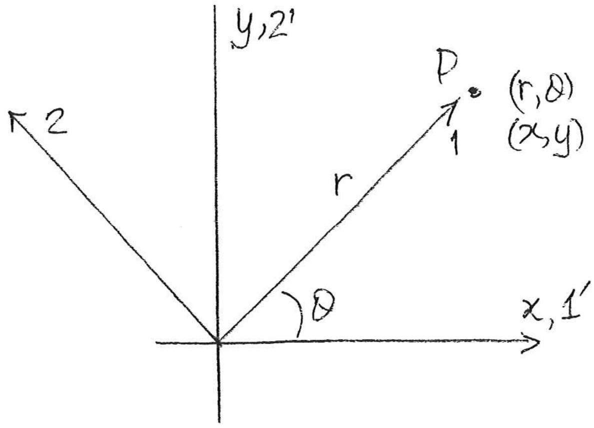
(ii) If friction forces act between A and B, assume that a line segment of A adheres to the die B and that sliding takes place on a line parallel and close to this surface. Additional term on right hand side is $k\, 2\sqrt{2}b . \sqrt{2}b$. This gives a minimum force at $x = 2b$ and $2F_{min} = 24kb$.

Q2. (c) Slip line fields are orthogonal networks, as dictated by equilibrium considerations. They give a solution that is both kinematically viable (such as shown here) but also statically viable.

Examiner's comments Errors more commonly algebraic than conceptual. Part (c) often omitted altogether.

3/ (a) $\phi = -Dr\ln r\cos\theta$

from Table I



$$\begin{cases} \sigma_{rr} = -\dfrac{D\cos\theta}{r} \\[2ex] \sigma_{\theta\theta} = -\dfrac{D\cos\theta}{r} \\[2ex] \sigma_{r\theta} = -\dfrac{D\sin\theta}{r} \end{cases}$$

(b) Rotation matrix between (r,θ) & (x,y) axes
i.e. $(1,2)$ & $(1',2')$ is

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

now $\sigma'_{\alpha\beta} = a_{\alpha i}\, a_{\beta j}\, \sigma_{ij}$

hence $\sigma_{xx} = \sigma'_{11} = a_{11}a_{11}\sigma_{11} + a_{12}a_{11}\sigma_{21}$

$+ a_{11}a_{12}\sigma_{12} + a_{12}a_{12}\sigma_{22}$

$\Rightarrow \cos^2\theta\,\sigma_{rr} - 2\sin\theta\cos\theta\,\sigma_{r\theta} + \sin^2\theta\,\sigma_{\theta\theta}$

But since $\sigma_{rr} = \sigma_{\theta\theta} = -\dfrac{D\cos\theta}{r}$ & $\sigma_{r\theta} = -\dfrac{D\sin\theta}{r}$

$$\sigma_{xx} = -\frac{D\cos\theta}{r} + \frac{2D\sin^2\theta\cos\theta}{r}$$

$$= -\frac{D\cos\theta}{r}\left\{1 - 2\sin^2\theta\right\}$$

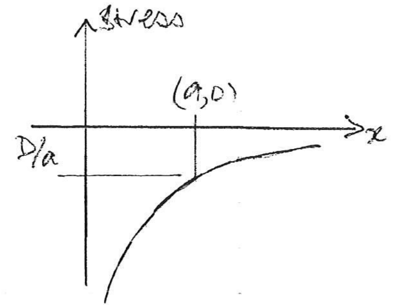
But $\cos\theta = x/r$; $\sin\theta = y/r$ & $x^2 + y^2 = r^2$

$\therefore \sigma_{xx} = -\dfrac{Dx}{r^2}\left\{1 - \dfrac{2y^2}{r^2}\right\}$ i.e. $\sigma_{xx} = \dfrac{-Dx(x^2-y^2)}{(x^2+y^2)^2}$

Similarly $\sigma'_{22} = a_{21}a_{21}\sigma_{11} + a_{22}a_{21}\sigma_{22} + a_{21}a_{22}\sigma_{12} + a_{22}a_{22}\sigma_{22}$

$$\Rightarrow \quad \sigma'_{yy} = \frac{-Dx(x^2-y^2)}{(x^2+y^2)^2}$$

and $\sigma'_{12} = \sigma_{xy} = \dfrac{Dy(x^2-y^2)}{(x^2+y^2)^2}$



along x axis $y=0$

$$\therefore \quad \sigma_{xx} = \sigma_{yy} = -\frac{D}{x} \; ; \quad \sigma_{xy} = 0$$

(c) from Table II

$$2Gu_\theta = \frac{D}{2}\left\{(K+1)\theta\cos\theta - \sin\theta - (K-1)\ln r \sin\theta\right\}$$

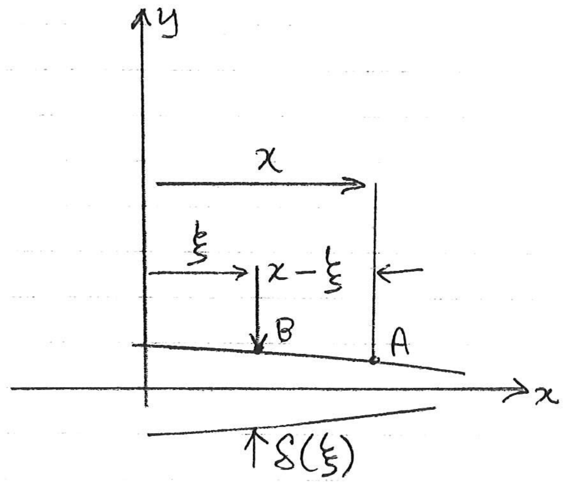
Thus if $\begin{cases} \theta = 0 & 2Gu_\theta = 0 \\ \theta = 2\pi & 2Gu_\theta = \dfrac{D}{2}\left\{(K+1).2\pi\right\} \end{cases}$

$$\text{Gap} \; \delta = \left|u_{\theta=0} - u_{\theta=2\pi}\right| = \frac{D(K+1)\pi}{2G}$$

In plane stress $K = \dfrac{3-\nu}{1+\nu}$ $\quad \therefore K+1 = \dfrac{3-\nu+1+\nu}{1+\nu} = \dfrac{4}{1+\nu}$

$$\therefore \delta = \frac{4D\pi}{2G(1+\nu)} = \frac{2\pi D}{G(1+\nu)}$$

(d)



Contribution to σ_{yy} at A from influence of dislocations between ξ and $(\xi + d\xi)$

will be $\dfrac{-D\,d\xi}{x-\xi}$

But $D = \dfrac{G(1+\nu).\delta(\xi)}{2\pi}$

$$\therefore \quad d\sigma_{yy} = -\frac{G(1+\nu)}{2\pi} \frac{\delta(\xi)\, d\xi}{(x-\xi)}$$

i.e. $$\sigma_{yy} = \frac{-G(1+\nu)}{2\pi} \int_{-a}^{+a} \frac{\delta(\xi)\, d\xi}{x-\xi}$$
for $-a<x<+a$

(not reqd) Over this region $\sigma_{yy} + S = 0$

$$\therefore \quad \int_{-a}^{a} \frac{\delta(\xi)}{x-\xi} d\xi = \frac{S.2\pi}{G(1+\nu)}$$

The solution to this equation is

$$\delta(\xi) = -\frac{2S\xi}{G(1+\nu)\sqrt{a^2-\xi^2}}$$

thus $$\sigma_{yy} = \frac{+\cancel{G(1+\nu)}}{\cancel{2}\pi} \cdot \frac{\cancel{2}S}{\cancel{G(1+\nu)}} \int_{-a}^{+a} \frac{\xi}{(x-\xi)\sqrt{a^2-\xi^2}} d\xi$$

(from tables of Integrals) $$= S\left\{-1 + \frac{|x|}{\sqrt{x^2-a^2}}\right\} \quad |x|>a$$

Now add uniform stress field $\sigma_{yy}=S$

to get: $$\sigma_{yy} = \frac{S|x|}{\sqrt{x^2-a^2}} \quad |x|>a$$

this tends to uniform field $\sigma_{yy}=S$ at $x\to\infty$ as it should and is singular at $x=\pm a$.

Examiner's comments: Few attempts as candidates apparently deterred by length & apparent complexity - but, in fact not difficult as found out by those who did attempt question which achieved high average mark.

JAW - April 10