PART IIB 2010 4F1 : CONTROL SYSTEM DESIGN

PROF M C SMITH

4F1 SOLUTIONS 2010

1 (a) $r(t) = 10$ arcmin s^{-1} $\Rightarrow$ $r(t) = 10t$

Steady-state error to ramp input $= 10/K_v$

where K_v = velocity error constant

$$= \lim_{s \to 0} s\,G(s)K(s) = K(0)$$

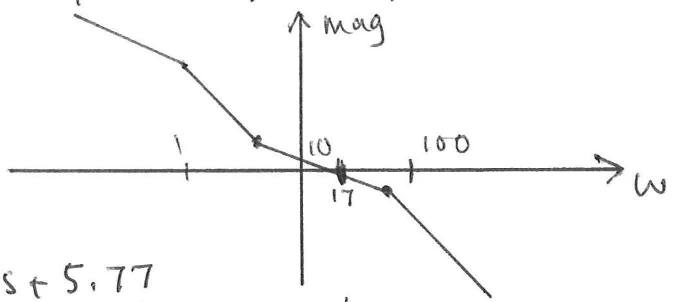
Spec A $\Rightarrow$ $K(0) \geq 100$

If we put $K(s) = 100$ the crossover will be around $\omega = 10$ rad/s and phase of $G(j\omega)$ is around -180°. Need a lead compensator.

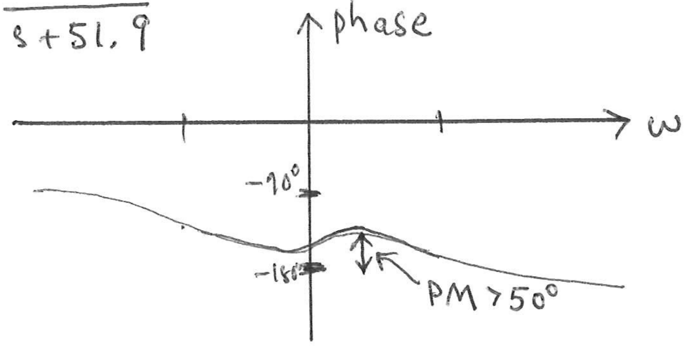
Choose

$$K(s) = k \cdot 3 \, \frac{s + w_c/3}{s + 3w_c}$$

which gives 53° max. phase lead at w_c and $k = 300$ to meet spec A. Around crossover freq. $|G(j\omega)| \sim 1/\omega^2 \Rightarrow k/w_c^2 \simeq 1 \Rightarrow w_c = 17.3$



$$K(s) = 900 \, \frac{s + 5.77}{s + 51.9}$$



(b) $G(s)$ has 2^{nd} order roll-off at high frequency and has no RHP poles. Hence

$$\int_0^\infty \ln|S(j\omega)|\,d\omega = 0$$

Spec E $\Rightarrow$ $|S(j\omega)| \leq \dfrac{1}{1-100\omega^{-2}}$ for $\omega \geq 100$

Hence:

$$0 \leq \int_0^{\omega_1} \ln 0.1\,d\omega + \int_{\omega_1}^{100} \ln\sqrt{2}\,d\omega - \int_{100}^{\infty} \ln(1-100\omega^{-2})\,d\omega$$

$$= \omega_1(-2.30) + (100-\omega_1)\,0.347 + 100\int_{100}^{\infty}\omega^{-2}\,d\omega$$

$$100\left[-\frac{1}{\omega}\right]_{100}^{\infty} = 1$$

$\Rightarrow$ $2.647\omega_1 \leq 35.7$

$\Rightarrow$ $\omega_1 \leq 13.49$ rad/sec

(c) At $\omega \geq 100$, $|G(j\omega)| \simeq \dfrac{1}{\omega^2}$. $k(j100) \simeq 900$ for large ω. Thus $|G(j\omega)k(j\omega)|$ is approximately 9 times too large at high frequency.

A lag compensator will be need to act above crossover to reduce the gain, but care must be taken not to worsen the PM too much.

Alternative approach to 1(a).

Crossover freq. for G is a little below 1 rad s^{-1} and $\angle G(j1) = -135°$. (In fact, $|G(j0.786)| = 1$ and $\angle G(j0.786) = -128.2°$.) So with $k(s) = 1$, PM $= 51.8°$, i.e. spec B is satisfied. We can now choose a $k(s)$ to leave $G(s)$ undisturbed at $\omega = 0.786$ and satisfy spec A: either with a lag compensator or proportional-plus-integral. Choosing

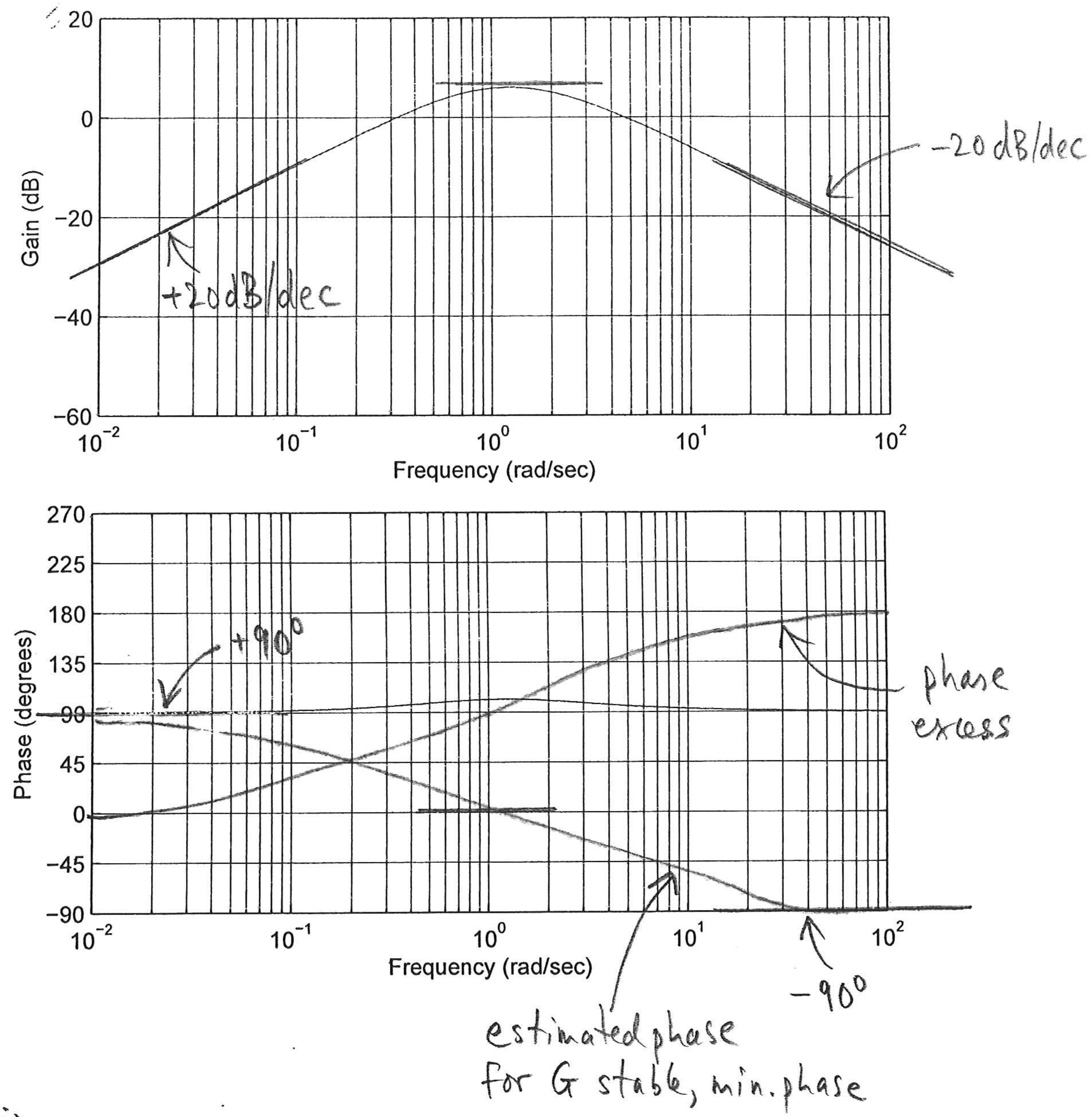
$$k(s) = \frac{s + 0.01}{s}$$

is such a solution. This solution also satisfies spec E – so no further modification is needed to get full marks for part (c).

(Aside remark: from the point of view of reducing sensitivity over as wide a frequency range as possible, the first solution with a lead compensator is better. It also has a better tracking transient response.)

Examiner's comment. Part (a) required spec A to be to be translated to a condition on the velocity error constant. Many candidates did this incorrectly, which affected the rest of the question. Many candidates successfully designed a suitable lead compensator to meet the specs. Part (b) was generally well done with most candidates knowing the correct procedure even if the calculation didn't always get to a correct value for the bound. Part (c) was done correctly by relatively few candidates.

2 (a)(i)

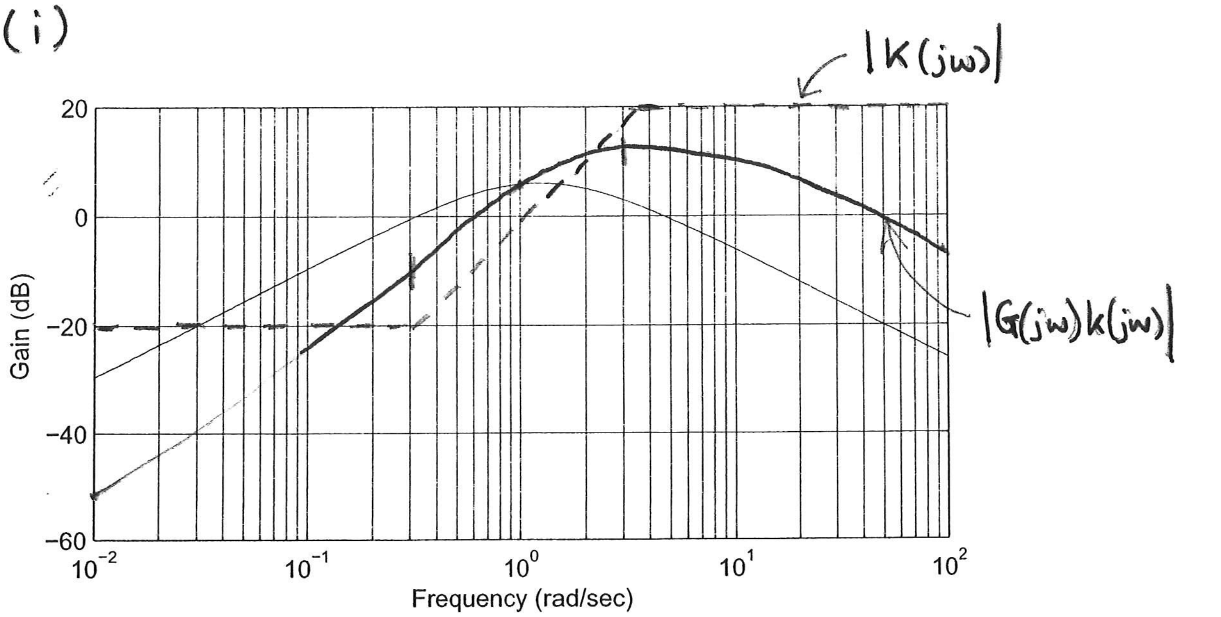


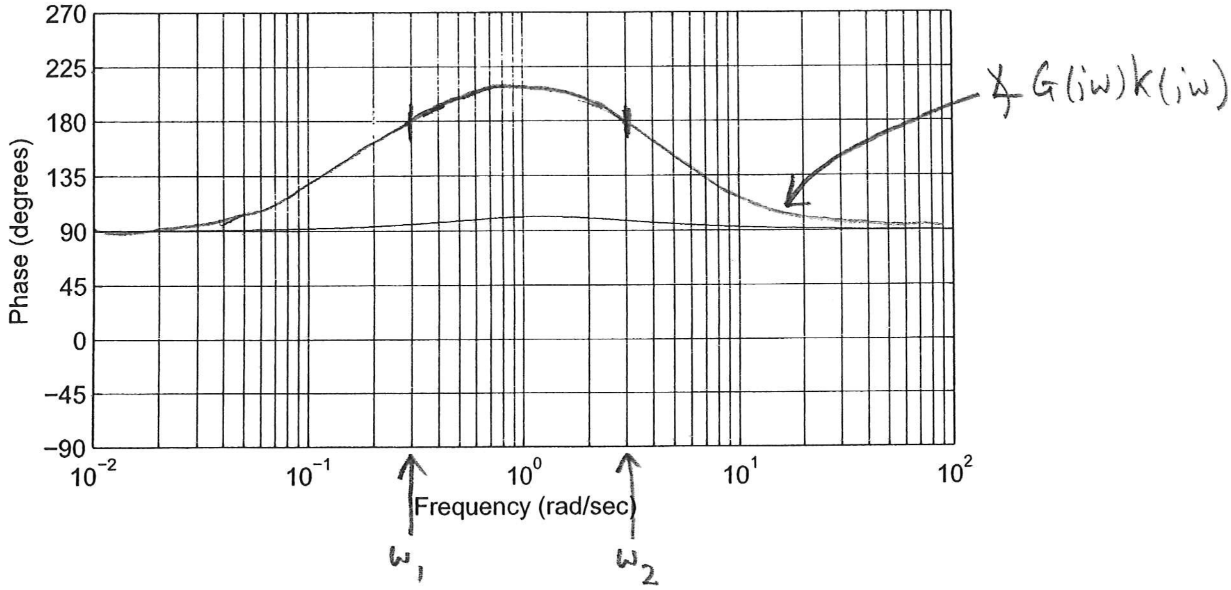
(ii)

Phase excess goes from 0° to +180° which is consistent with one RHP pole and no RHP zeros.

But, the positive slope of 20 dB/dec at low frequencies suggests a zero at s = 0.

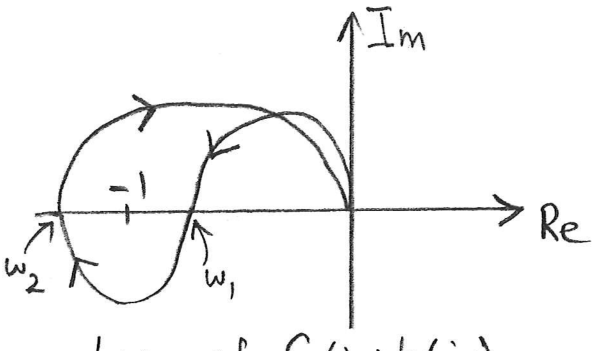
2(b)(i)





(ii) Notice that $|k(jw)| < 1$ for $w < 1$ and $|k(jw)| > 1$ for $w > 1$. There are two frequencies $w_1 < w_2$ where $\angle(G(jw)k(jw)) = 180^\circ$. So $|G(jw_1)k(jw_1)| < 1$ and $|G(jw_2)k(jw_2)| > 1$.

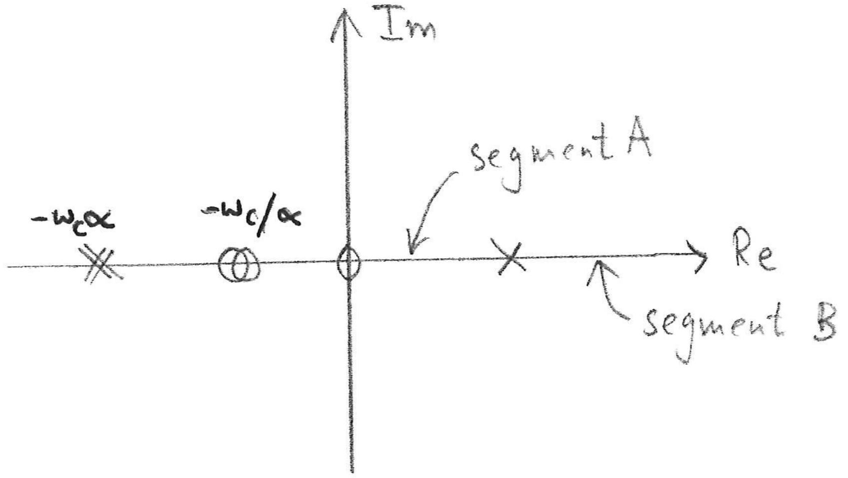
Positive frequency part of Nyquist diagram is shown:



Locus of $G(jw)k(jw)$ for $w > 0$.

Complete Nyquist diagram has two clockwise or no encirclements about $-1/k$ for any k. One counterclockwise encirclement needed for closed-loop stability ⇒ can never be closed-loop stable.

2(c)(i) The known poles and zeros of $G(s)k(s)$ appear as follows in the complex plane:

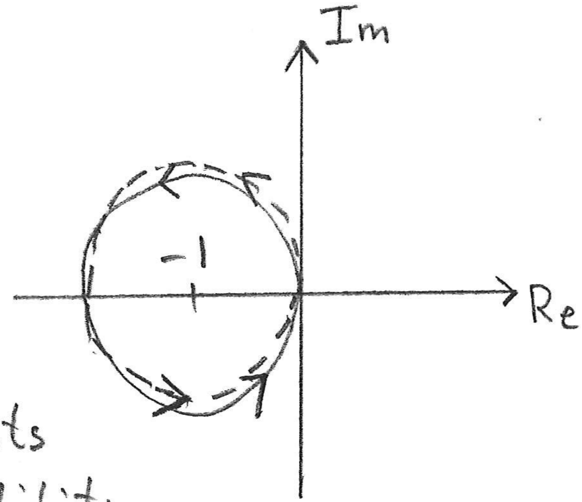


There may be other poles and zeros, but not in the RHP. For either choice of sign of feedback gain segment A or segment B is on the root-locus, so the system cannot be stabilised.

(c)(ii) Choose $k(s) = \dfrac{1+s}{1-s}$.

From Bode diagram (see overleaf) the Nyquist diagram takes the following form:

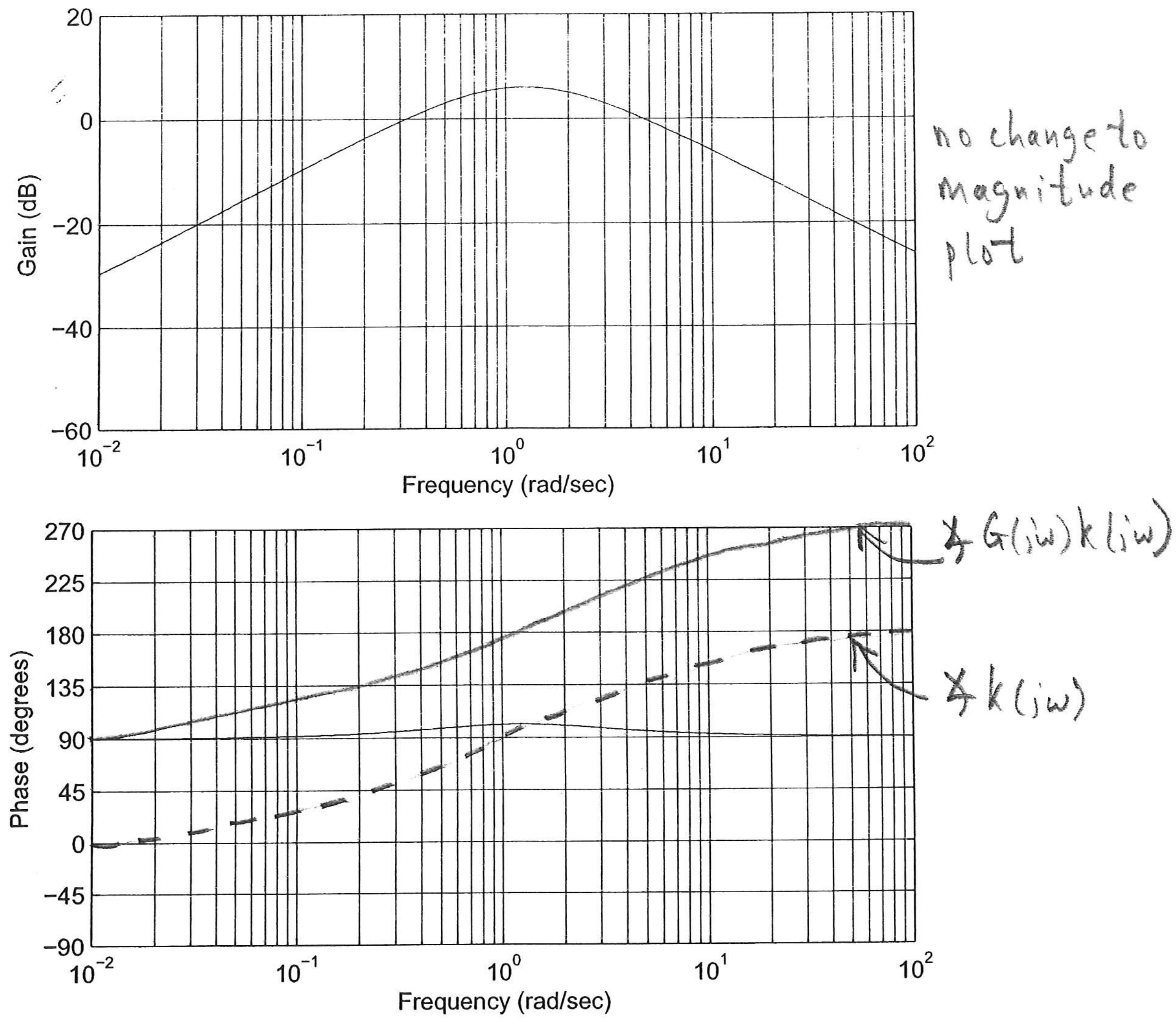
Note: the positive and negative frequencies each give a counter-clockwise loop around the -1 point.



2 counterclockwise encirclements about -1 $\Rightarrow$ closed-loop stability

(2 poles in RHP in open-loop, one from G and one from k).

2 (c)(ii) (cont.)



Examiner's comment. Part (a) was fairly standard and correctly executed by most candidates, however, many did not spot that there is a zero at $s = 0$. In part (b) most candidates produced a correct Bode diagram but then had difficulty drawing the corresponding Nyquist diagram. In part (c), candidates who had correctly identified the zero at $s = 0$ were able to give a good answer to (i). A number of candidates failed to correctly apply the Nyquist stability criterion in part (ii), however, there were many good answers.

3 (a) RHP zeros cannot be removed and the high-frequency roll-off rate cannot be decreased.

(b) (i) $\bar{y}(s) = G(s)\frac{1}{s}$ DC gain $= 1$

$$\dot{y}(0^+) = \lim_{s\to\infty} s\,\mathcal{L}(\dot{y}(t))$$
$$= \lim_{s\to\infty} s^2\bar{y}(s) = \lim_{s\to\infty} s\,G(s)$$
$$= \prod_{i=1}^{n-1}\alpha_i \Big/ \prod_{i=1}^{n}\beta_i$$

initial undershoot $\Leftrightarrow \dot{y}(0^+) < 0$

$\Leftrightarrow$ odd number of α_i negative

since all β_i are positive.

(ii) $\bar{y}(-1/\alpha_k) = 0$

$$\bar{y}(s) = \int_0^\infty y(t)\,e^{-st}\,dt$$

Since $\alpha_k < 0$, $-1/\alpha_k$ is in the region of convergence of the transform. Hence

$$0 = \int_0^\infty y(t)\,e^{+t/\alpha_k}\,dt$$

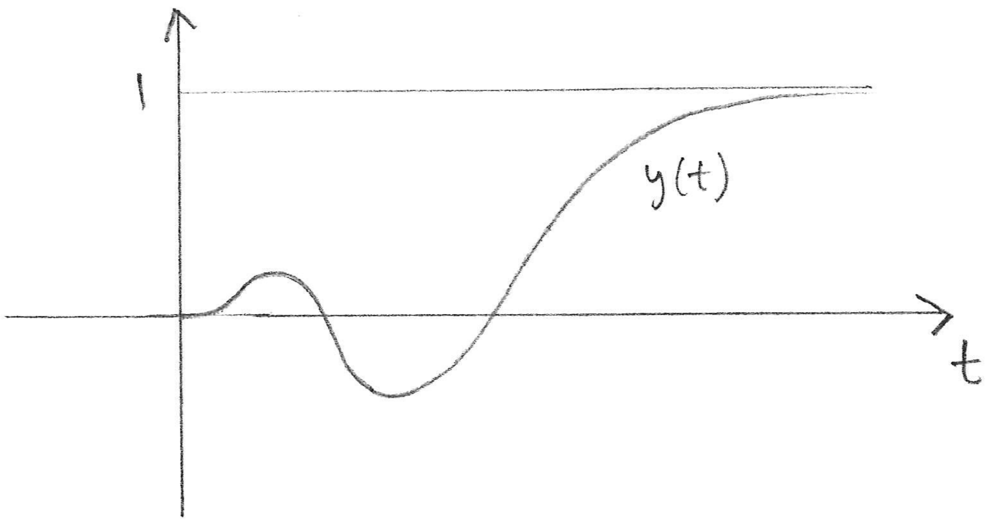
(iii) The integral in (ii) implies that $y(t)$ will be positive at some times and negative at others. Hence there must be at least one $t_1 > 0$ where $y(t_1) = 0$

(c)(i)

$$\dot{y}(0^+) = \lim_{s\to\infty} s\,G(s) = 0$$

$$\ddot{y}(0^+) = \lim_{s\to\infty} s^2 G(s) = 1$$

(ii)



Initial slope zero, but moving up due to positive 2nd derivative. RHP zero(s) implies output must go to zero at some time.

(Further calculation, not needed, to show lack of overshoot.)

(c) (iii) Output must go to (or more likely through) zero at some positive time. Initial slope of step response must always be zero.

Examiner's comment. Part (a) was straightforward bookwork which most candidates wrote down correctly. Part (b) was solved correctly by a majority. In part (c) candidates often failed to apply the initial value theorem correctly and also to draw the correct conclusions from the derivations of part (b). Nevertheless, a good proportion of candidates had grasped the essential points.