PART IIB 2010 4F2: ROBUST MULTIVARIABLE CONTROL

DR J GONCALVES

1 (a) The relationship is:

$$\|G(s)\|_\infty = \sup\{\|u\|_2 : \|w\|_2 \le 1\}.$$

[10%]

(b) We have that

$$V(x_\infty) - V(x_0) = \int_0^\infty \frac{dV}{dt} dt \le \int_0^\infty \left(\gamma^2 w^T w - u^T u\right) dt.$$

Since V is a Lyapunov function, $V(x_\infty) \le V(x_0) = 0$. Hence, if $\|w\|_2 > 0$, we get:

$$\frac{\|u\|_2}{\|w\|_2} \le \gamma$$

[25%]

(c) We have that

$$\begin{aligned}
\frac{dV}{dt} + u^T u - \gamma^2 w^T w = {} & x^T M(Ax + w + u) \\
& + \left(u^T + w^T + x^T A^T\right) Mx + u^T u - \gamma^2 w^T w \\
= {} & x^T \left(MA + A^T M\right) x \\
& + u^T u + u^T Mx + x^T Mu - \gamma^2 w^T w + w^T Mx + x^T Mw \\
= {} & x^T \left(MA + A^T M - M^2 + \frac{1}{\gamma^2} M^2\right) x \\
& + (u + Mx)^T (u + Mx) \\
& - \gamma^2 \left(w - \frac{1}{\gamma^2} Mx\right)^T \left(w - \frac{1}{\gamma^2} Mx\right).
\end{aligned}$$

The RHS is non-positive if we set: $u = -Mx$, and,

$$MA + A^T M - M^2 + \frac{1}{\gamma^2} M^2 = 0.$$

We still need the closed loop system to be stable. This requirement is satisfied if the matrix $A - M$ is stable. [40%]

(d) M is a scalar because x is scalar. C1 implies that:

$$0 = 2aM - M^2 + \frac{1}{\gamma^2} M^2$$

$$0 = M\left(\frac{2a}{1 - \frac{1}{\gamma^2}} - M\right).$$

(cont.

Hence, for a positive value for M, we need:

$$\gamma^2 > 1.$$

Condition C2 implies that

$$1 - \frac{2a}{1 - \frac{1}{\gamma^2}} < 0.$$

This also implies that $\gamma^2 > 1$. Since γ is positive, we get $\gamma > 1$. [25%]

Assessor's comment:

This was the question that was the least attempted. A straightforward question, well answered by most candidates, which resulted in the highest mean of all three questions. The only difficulty was in the last part, proving a lower bound that could be obtained directly from the given conditions.

(TURN OVER

2 (a) For $k = 1, 2, \ldots, h-1$,

$$V_k(x) = \inf_{u_k} \{V_{k+1}(f(x, u_k)) + c(x, u_k)\},$$

where the value function

$$V_k(x) = \inf_{(u_k, \ldots, u_{h-1})} \left\{ \sum_{i=k}^{h-1} c(x_i, u_i) + J_h(x_h) \right\},$$

where $x_k, x_{k+1}, \ldots, x_h$ is the sequence of states generated by $u_k, u_{k+1}, \ldots, u_{h-1}$ starting with $x_k = x$. [30%]

(b) The value function at time h is $J_h(x_h)$. [10%]

(c) (i) The dynamic programming equations for choosing u_k optimally for $k = 1, 2, \ldots, h-1$ are:

$$V_k(v) = \inf_u \left\{ V_{k+1}(v-u) + \frac{u^2}{R_k} \right\},$$

[10%]

(ii) Clearly, V_h is quadratic. We have:

$$\begin{aligned} V_{h-1}(v) &= \inf_u \left\{ \frac{u^2}{R_{h-1}} + \frac{(v-u)^2}{R_h} \right\} \qquad (3) \\ &= \frac{v^2}{R_{h-1} + R_h}. \end{aligned}$$

where we have used the result of the hint on the last equality. Hence, $V_{h-1}(v)$ is also quadratic and of the form $\frac{v^2}{C_{h-1}}$, where C_{h-1} is a positive constant. In fact, C_{h-1} equals $R_h + R_{h-1}$. We now assume that V_{k+1} is a quadratic function of the form $\frac{v^2}{C_{k+1}}$, and show that this implies V_k is also quadratic

(cont.

with the form: $\frac{v^2}{R_k+C_{k+1}}$:

$$\begin{aligned}
V_k(v) &= \inf_u \left\{ \frac{u^2}{R_k} + \frac{(v-u)^2}{C_{k+1}} \right\} \\
&= \inf_u \left\{ u^2 \left[\frac{1}{R_k} + \frac{1}{C_{k+1}}\right] - 2u \frac{\sqrt{\left[\frac{1}{R_k} + \frac{1}{C_{k+1}}\right]}}{\sqrt{\left[\frac{1}{R_k} + \frac{1}{C_{k+1}}\right]}} \frac{v}{C_{k+1}} + \frac{v^2}{C_{k+1}} \right\} \qquad (4) \\
&= \frac{v^2}{C_{k+1}+R_k}.
\end{aligned}$$

By induction, we have that all the value functions are quadratic. Moreover, $C_k = \sum_{i=k}^{h} R_i$. [30%]

(iii) Using again the result of the hint, equation (4) implies that the minimising choice is:

$$\begin{aligned}
u_k^* &= v_k \times \frac{1}{\left[\frac{1}{R_k} + \frac{1}{C_{k+1}}\right] \times C_{k+1}} \\
&= v_k \times \frac{R_k}{R_k + C_{k+1}} \\
&= v_k \times \frac{R_k}{R_k + R_{k+1} + \cdots + R_h}.
\end{aligned}$$

which gives the same result as the one obtained by applying Ohm's and Kirchoff's laws. [20%]

Assessor's comment:

This was the question with the lowest average. The first parts were straightforward. However, most students failed to solve the dynamic programming equations and even just show that a quadratic value function would solve the problem.

(TURN OVER

3 (a) A feedback interconnection between G and Δ is stable for all G satisfying $G \in H_\infty$, $\|G\|_\infty < 1$ if, and only if, $\Delta \in H_\infty$ and $\|\Delta\|_\infty \leq 1$.

H_∞ is the space of transfer function matrices satisfying $\bar{\sigma}(G(s)) < \infty, \forall s: \text{Re}(s) > 0$. For $G \in H_\infty$, $\|G\|_\infty := \sup_{s:\, Re(s)>0} \bar{\sigma}(G(s)) = \sup_\omega \bar{\sigma}(G(j\omega))$. [20%]

(b) Let $G_1 = (I + W\Delta)^{-1}G$. This may be represented as as $y = \Delta W y + Gu$, or

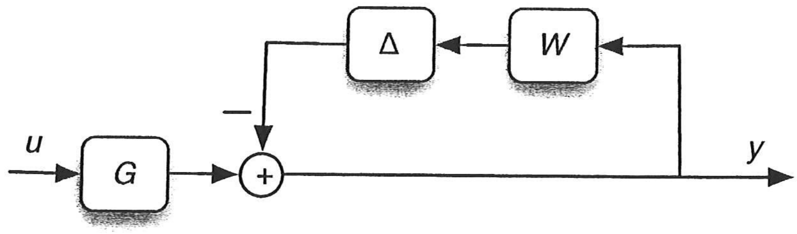

Fig. 2

$(I + \Delta W)y = Gu$ which means that $y = (I + \Delta W)^{-1}Gu$ as required. Adding a controller gives

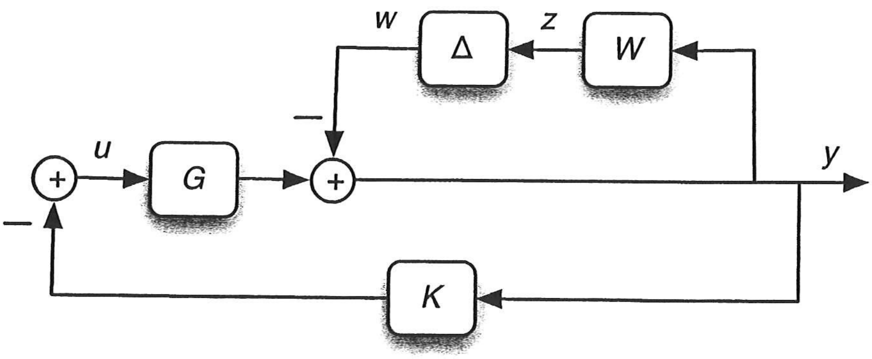

Fig. 3

Now, $y = -GKy - w$ or $y = -(I + GK)^{-1}w$. Since $z = Wy$, then $z = -W(I + GK)^{-1}w$. Using the Small Gain Theorem, we see that the feedback system is stable for all Δ, $\|\Delta\|_\infty < \varepsilon$ if, and only if, $\|W(I + GK)^{-1}\|_\infty \leq 1/\varepsilon$.

(cont.

(i) Since $G_1 = G/(1+W\Delta)$, then

$$W\Delta = \frac{G}{G_1} - 1 = \frac{\delta_1 s + \delta_2}{s^2+2s+0.2}$$

Let

$$W = \frac{s+1}{s^2+2s+0.2}$$

which means that

$$\Delta = \frac{\delta_1 s + \delta_2}{s+1}$$

Hence, $\|\Delta\|_\infty < 1$ if $|\delta_1| < 1$ and $|\delta_2| < 1$.

Now $K = 0.8$ which means that

$$\frac{1}{1+GK} = \frac{s^2+s+0.2}{s^2+2s+1}$$

and

$$\frac{W}{1+GK} = \frac{s+1}{s^2+2s+0.2}\,\frac{s^2+s+0.2}{s^2+2s+1} = \frac{1}{s+1}$$

Therefore, $\|W(I+GK)^{-1}\|_\infty \leq 1$ and the system is stable using the result from part (b).

(ii) The closed loop poles are the roots of $s^2+(2+\delta_1)s+1+\delta_2 = 0$. For close loop stability we then require $\delta_1 > -2$ and $\delta_2 > -1$. So, Small Gain Theorem is conservative in this case The results can be improved using μ, writing the close loop as in Fig. 4 and checking $\mu(M)$.

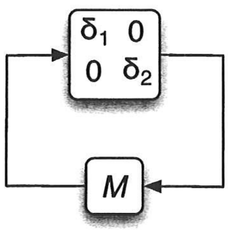

Fig. 4

Assessor's comment:

This question was answered by all but 1 student. Overall a straightforward question, well answered by most candidates. For part (c)(ii) most students did not explain how to use mu to reduce conservatism.