Module 4F3: Nonlinear and Predictive Control
Solutions 2010

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1. (a) i. A set S is invariant if for all $x(0) \in S$ and for all $t \geq 0$, $x(t) \in S$. Examples: equilibrium points, limit cycles

   ii. An equilibrium x_e is stable if $\forall \epsilon > 0 \ \exists \delta > 0$ s.t. $\|x(0) - x_e\| < \delta$ implies $\|x(t) - x_e\| < \epsilon \ \ \forall\, t \geq 0$.

   iii. An equilibrium x_e is asymptotically stable if it is stable and $\exists \delta > 0$ s.t. $\|x(0) - x_e\| < \delta$ implies $\lim_{t \to \infty} x(t) = x_e$.

   iv. An equilibrium x_e is globally asymptotically stable if it is stable and $\forall \delta > 0$, $\|x(0) - x_e\| < \delta$ implies $\lim_{t \to \infty} x(t) = x_e$.

   (b) LaSalle's Theorem. Let S be a compact invariant set. Assume there exists a differentiable function $V : S \to \mathbb{R}$ s.t.

$$\dot{V}(x) \leq 0 \ \ \forall x \in S$$

   Let M be the largest invariant set contained in $\{x \in S : \dot{V}(x) = 0\}$. Then $\forall x(0) \in S$, $x(t) \to M$ as $t \to \infty$.

   Main advantages: Can be used to prove convergence to an invariant set, can be used to determine regions of attraction, can be technically more convenient since $\dot{V}$ is required to be non-positive rather than strictly negative.

   (c) i. First verify that $x = 0$ is an equilibrium point. Note that this is unique.

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial z}\dot{z} + \sum_{i=1}^{m} \frac{\partial V}{\partial y_i}\dot{y}_i \\
&= 2\alpha z\left(-\sum_{i=1}^{m} f_i(y_i)\right) + \sum_{i=1}^{m} f_i(y_i)(-h(z,y)y_i + z) \\
&= -\sum_{i=1}^{m} y_i f_i(y_i) h(z,y) + (1-2\alpha) z \sum_{i=1}^{m} f_i(y_i) \\
&= -\sum_{i=1}^{m} y_i f_i(y_i) h(z,y) \qquad \textbf{choosing } \alpha = 1/2 \\
&\leq 0
\end{aligned}$$

   Choose $c > 0$ and define $S := \{(z,y) : V(z,y) \leq c\}$. S is a closed and bounded invariant set, hence from LaSalle's theorem for any $x(0) := (z(0), y(0))$ s.t. $x(0) \in S$, $x(t) \to M$ as $t \to \infty$, where M is the largest invariant set in S included in $\{(z,y) : \dot{V}(z,y) = 0\}$.

$$\dot{V}(t) = 0 \Rightarrow y_i(t) = 0 \quad \forall i$$

If $z(t) \neq 0$ then $\dot{y}_i(t) \neq 0$, hence $\exists \tau$ s.t. $y_i(t+\tau) \neq 0$, therefore M only includes the origin.

ii. The Jacobian is given at the origin by

$$\begin{bmatrix} 0 & \frac{\partial f_1}{\partial y_1} & \dots & \frac{\partial f_m}{\partial y_m} \\ 1 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}$$

For $f_i(y_i) = y_i^3$ there is at least one eigenvalue at the origin, since the first row contains only zeros (hence matrix is singular, and determinant is product of eigenvalues, or note that λ is a factor of the characteristic polynomial,...) $\Rightarrow$ linearization inconclusive.

Assessor's comment:

This question first asked for some definitions and a statement of LaSalle's theorem. It then asked for the application of Lyapunov methods to prove the stability of a fairly complicated system, using a Lyapunov function of a kind that was presented in the course. The great majority of candidates knew what they wanted to do, but surprisingly many had trouble partially differentiating an expression of the form $\sum_i f_i(y_i)$.

2. (a) Consider $e = E\sin\theta$. If $E \le \delta$. $f(e) = 0$, hence $N_\delta(E) = 0$.
If $E > \delta$ then

$$N_\delta(E) = \frac{U_1 + jV_1}{E}$$

$V_1 = 0$ since $f(e)$ is an odd function.

$$\begin{aligned}
U_1 &= \frac{1}{\pi}\int_0^{2\pi} f(E\sin\theta)\sin\theta d\theta = \frac{4}{\pi}\int_0^{\pi/2} f(E\sin\theta)\sin\theta d\theta \\
&= \frac{4}{\pi}\int_{\sin^{-1}(\delta/E)}^{\pi/2} (E\sin\theta - \delta)\sin\theta d\theta \\
&= \frac{4E}{\pi}\int_{\sin^{-1}(\delta/E)}^{\pi/2} \left[\frac{1-\cos(2\theta)}{2} - \frac{\delta}{E}\sin\theta\right] d\theta \\
&= \frac{2E}{\pi}\left[\theta - \frac{\sin(2\theta)}{2} + \frac{2\delta}{E}\cos\theta\right]_{\sin^{-1}(\delta/E)}^{\pi/2} \\
&= \frac{2E}{\pi}\left[\frac{\pi}{2} - \sin^{-1}(\delta/E) + \frac{1}{2}\sin(2\sin^{-1}(\delta/E)) - \frac{2\delta}{E}\cos(\sin^{-1}(\delta/E))\right] \\
&= E - \frac{2E}{\pi}\left[\sin^{-1}(\delta/E) - \frac{\delta}{E}\cos(\sin^{-1}(\delta/E)) + \frac{2\delta}{E}\cos(\sin^{-1}(\delta/E))\right] \\
&= E - \frac{2E}{\pi}\left[\sin^{-1}(\delta/E) + \frac{\delta}{E}\sqrt{1-(\delta/E)^2}\right]
\end{aligned}$$

(b) We have $g(e) = f_1(e) - f_2(e)$ where the subscript in f denotes the value of δ. So $N(E) = N_1(E) - N_2(E)$

(c) i. We have the sector bound $0 \le ef(e) \le e^2$. So for stability we need $\Re[G(j\omega)] > -1$.

$$G(j\omega) = k\frac{j\omega}{(2+j\omega)^2} = k\frac{j\omega(2-j\omega)^2}{(4+\omega^2)^2}$$

So

$$\Re[G(j\omega)] = \frac{4k\omega^2}{(4+\omega^2)^2}$$

To find the maximum value of $\Re[\frac{1}{k}G(j\omega)]$ we set

$$\frac{\partial\Re[\frac{1}{k}G(j\omega)]}{\partial\omega^2} = \frac{4(4+\omega^2)^2 - 2(4+\omega^2)4\omega^2}{(4+\omega^2)^4} = 0 \Rightarrow (4+\omega^2) - 2\omega^2 = 0 \Rightarrow \omega^2 = 4$$

We therefore have

$$0 \le \frac{4\omega^2}{(4+\omega^2)^2} < \frac{1}{4}$$

And hence need $k > -4$ to guarantee stability by the circle criterion.

ii. No. We have from the sector bound that $0 < N_\delta(E) < 1$. Hence $-1/N_\delta(E)$ does not intersect $G(j\omega)$, and therefore no limit cycles are predicted.

iii. Yes. $g(e)$ satisfies the same sector bound.

Assessor's comment:

This was a rather standard question on describing functions, on using the circle criterion, and on comparing the two. Some candidates seemed to write down integrals from memory rather than deduce them from definitions, which was ok except for those who mis-remembered them. There was the usual set of candidates who did not know how to sketch a Nyquist locus, or did not attempt to do so.

3. (a)

$$x_{s+1} = Ax_s + Bu_s \tag{1}$$

Thus we get

$$\begin{aligned} x_1 &= Ax_0 + Bu_0 & (2)\\ x_2 &= Ax_1 + Bu_1 & (3)\\ &= A[Ax_0 + Bu_0] + Bu_1 & (4)\\ &= A^2x_0 + ABu_0 + Bu_1 & (5) \end{aligned}$$

Thus we have $X = \Phi x_0 + \Gamma U$ if:

$$\Phi = \begin{bmatrix} A \\ A^2 \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} B & 0 \\ AB & B \end{bmatrix} \tag{6}$$

(b) The given constraints are

$$Cu_0 \leq e, \quad Cu_1 \leq e, \quad Dx_1 \leq f. \tag{7}$$

These three constraints can be written in the form

$$FX + GU \leq g \tag{8}$$

if F, G, g each has 3 rows (one for each constraint), and are defined as follows:

$$F = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ D & 0 \end{bmatrix}, \quad G = \begin{bmatrix} C & 0 \\ 0 & C \\ 0 & 0 \end{bmatrix}, \quad g = \begin{bmatrix} e \\ e \\ f \end{bmatrix}. \tag{9}$$

Note: The solution to this part is not unique, but the idea is to translate the constraints into the form of (8) as simply as possible.

(c) Now we need to eliminate the 'FX' term from (8). From part (a) we have $X = \Phi x_0 + \Gamma U$. Substituting into (8) gives

$$F(\Phi x_0 + \Gamma U) + GU \leq g \tag{10}$$

or

$$(F\Gamma + G)U \leq g - F\Phi x_0 \tag{11}$$

hence we have

$$S = F\Gamma + G, \quad T = -F\Phi, \quad h = g \tag{12}$$

(11) is a linear inequality constraint in the decision variables (which are the elements of U). Optimisation of a convex cost function with such a linear inequality constraint gives a convex optimisation problem. (Equally, (8) is a linear inequality constraint in the decision variables, if the elements of X are left in the problem instead of being eliminated in step (c).)

(d) Convex problems have the important property that a search for the minimum in a 'downhill' direction is guaranteed to terminate at a global minimum. Also the time required to perform such a search is very predictable in practice. Both of these are important features, since the control signal is found in MPC by solving optimisation problems in real time.

(e) An absolute value component of the cost is appropriate if one wants to minimise, for example, the amount of fuel used in a spacecraft, or the amount of chemical added during paper-making. Quadratic costs on the input are more appropriate, however, if one wants to minimise the energy used, since that varies typically as the square of electric current, speed of a momentum wheel, etc.

Assessor's comment:

This was a very standard question on predictive control. The final part, on alternative cost functions, required candidates to have read slightly beyond the lectures, or to use their engineering insight. It was answered very well by a small proportion of candidates. Overall this question was rather too easy.

4. (a) 'Setpoint tracking' is the ability of a control system to make the controlled outputs follow their setpoint values to within some specification. Often the setpoint is piecewise-constant, but it can also vary with time in some applications.

A target calculator is used with an MPC regulator as shown in Fig.1 to obtain setpoint tracking. The MPC regulator attempts to drive $x(k)$ to x_∞ and $u(k)$ to u_∞. This is done by minimising a cost function such as

$$\sum_{i=0}^{N} \|x_{k+i} - x_\infty\|_Q^2 + \|u_{k+i} - u_\infty\|_R^2 \tag{13}$$

where x_{k+i} and u_{k+i} are the i-step ahead predictions of x and u.

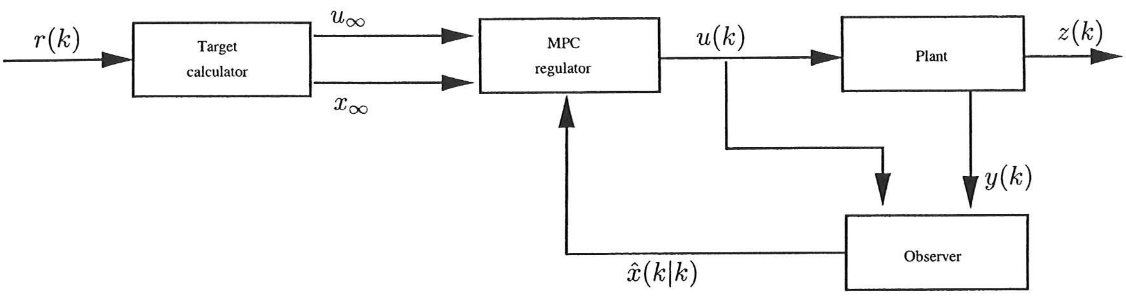

Figure 1: Block diagram for Q.4(a)

(b) i. An offset-free target-equilibrium pair is a pair (x_∞, u_∞) which gives a desired reference value r for the controlled outputs in steady state:

$$x_\infty = Ax_\infty + Bu_\infty \tag{14}$$
$$Hx_\infty = r \tag{15}$$

ii. This pair of equations can be rewritten as the single equation

$$\begin{bmatrix} I-A & -B \\ H & 0 \end{bmatrix} \begin{bmatrix} x_\infty \\ u_\infty \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix} \tag{16}$$

So the target-equilibrium pair exists if this equation has a solution. But it has a solution if the matrix on the left has full row-rank (because then the number of variables is at least as big as the number of equations).

iii. A necessary condition for the matrix to have full row-rank is that the number of rows should not exceed the number of columns. But the number of rows is $n_x + n_z$, where n_x is the number of states and n_z is the number of controlled variables, and the number of columns is $n_x + n_u$, where n_u is the number of inputs. Thus $n_z \le n_u$ is a necessary condition for the matrix to have full row-rank.

(c) If a disturbance acts as given in the question, then it is necessary to take the steady-state effect of the disturbance into account when calculating the target-equlibrium pair:

$$x_\infty = Ax_\infty + Bu_\infty + B_d d \tag{17}$$
$$Hx_\infty = r \tag{18}$$

Furthermore, it is necessary to estimate d in order to solve these equations for (x_∞, u_∞). To do this an observer can be used, with the augmented state $[x(k), d(k)]$, then the estimate $\hat{d}(k|k)$ is used instead of d in (17).

(d) If constraints are imposed on the states and inputs then there may not be a solution to (17)–(18) which is consistent with the constraints. In that case one possibility is to find a pair (x_∞, u_∞) which is feasible and is as close to solving (17)–(18) as possible, eg in a least-squares sense. Thus one solves a problem such as

$$[x_\infty^*, u_\infty^*] = \arg\min_{x,u} \|(I-A)x - Bu - B_d\hat{d}(k|k)\|^2 + \|Hx - r\|^2 \tag{19}$$

subject to the constraints being satisfied. (Other optimisation problems can also be formulated for this purpose.)

Assessor's comment Q4:

This question focussed on setpoint tracking in predictive control, a subject which has not come up much in previous years' exams. It was answered impressively well by the small number of candidates that attempted it.

Module 4F3: Nonlinear and Predictive Control
Answers to 2010 exam.

1. (c)(ii) No it could not.
2. (c)(i) $k > -4$. (ii) No. (iii) Yes.