PART IIB 2010 DR A GUILLÉN i FÀBREGAS

4F5 Advanced Wireless Communications, 2010
Crib

Question 1

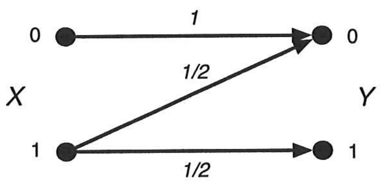


Figure 1: Channel model for Question 1 (a).

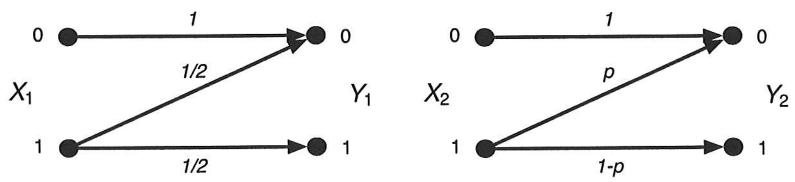


Figure 2: Channel model for Question 1 (b).

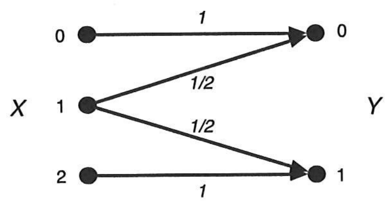


Figure 3: Channel model for Question 1 (c).

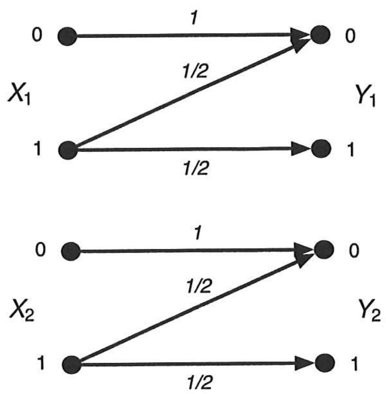


Figure 4: Channel model for Question 1 (d).

(a) *Consider the channel described in Fig. 1. This channel is a good model for optical fibre transmission; when no photon is transmitted, with probability* 1, *no photon will be received; when a photon is transmitted it can be lost with a certain probability. Calculate* $H(Y|X), H(X,Y), H(Y), H(X|Y)$ *and* $I(X;Y)$ *for equiprobable input symbols.*

Equiprobable input symbols signifies $P_X(0) = P_X(1) = 1/2$. We compute the joint distribution values

$$P_{XY}(0,0) = P_X(0)P_{Y|X}(0|0) = \frac{1}{2}\cdot 1 = \frac{1}{2}$$
$$P_{XY}(0,1) = P_X(0)P_{Y|X}(1|0) = \frac{1}{2}\cdot 0 = 0$$
$$P_{XY}(1,0) = P_X(1)P_{Y|X}(0|1) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$
$$P_{XY}(1,1) = P_X(1)P_{Y|X}(1|1) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.$$

We obtain the output distribution by marginalising the joint distribution:

$$P_Y(0) = P_{XY}(0,0) + P_{XY}(1,0) = \frac{3}{4}$$
$$P_Y(1) = P_{XY}(0,1) + P_{XY}(1,1) = \frac{1}{4}$$

We are now ready to compute the quantities requested:

$$
\begin{aligned}
H(Y) &= h(1/4) = \frac{1}{4}\log_2 4 + \frac{3}{4}\log_2 \frac{4}{3} = 2 - \frac{3}{4}\log_2 3 = 0.8113 \\
H(X,Y) &= H(X) + H(Y|X) \\
&= H(X) + P_X(0)H(Y|X=0) + P_X(1)H(Y|X=1) \\
&= h(1/2) + \frac{1}{2}h(1) + \frac{1}{2}h(1/2) = \frac{3}{2} \\
H(X|Y) &= H(X,Y) - H(Y) = 0.6887 \\
H(Y|X) &= H(X,Y) - H(X) = 1.5 - 1 = 0.5 \\
I(X;Y) &= H(X) - H(X|Y) = 0.3113
\end{aligned}
$$

(b) *Let Y_1 be the output of channel 1 to input X_1 and Y_2 be the output of channel 2 to input X_2 (see Fig. 2). Find the value of p that makes $I(X_1;Y_2) = I(X_1;Y_1)$ when the output of channel 1 is used as input for channel 2, i.e., $Y_1 = X_2$. Is there any value of p that makes $I(X_1;Y_2) > I(X_1;Y_1)$? Justify your answer.*

The value $p = 0$ turns the second channel into a noiseless channel and thus

$$I(X_1;Y_2) = I(X_1;Y_1).$$

It is not possible to select p such that $I(X_1;Y_2) > I(X_1;Y_1)$ due to the data processing theorem, which states that mutual information over the outer variables in a Markov chain is at most equal to the mutual information over the inner variables.

(c) *Find the capacity and the capacity-achieving input distribution for the channel shown in Figure 3. Justify your answer.*

The input distribution $P_X(0) = P_X(2) = 1/2$ and $P_X(1) = 0$ gives a mutual information $I(X;Y) = 1$. The capacity of this channel satisfies

$$C = \max_{P_X} I(X;Y) = \max_{P_X}[H(Y) - H(Y|X)] \le \max_{P_X} H(Y) \le 1$$

where the last inequality follows from the fact that output variable Y is binary. Therefore, we conclude that the capacity of the channel is $C = 1$.

(d) *Let Y_1 be the output of channel 1 to input X_1 and Y_2 be the output of channel 2 to input X_2 (see Fig. 4. Consider the following transmission strategy: transmit a photon on channel 1 when a 0 is to be sent; transmit a photon on channel 2 when a 1 is to be sent. Draw the equivalent channel transition diagram and compare it to any of the known discrete memoryless channels studied in the notes. What is its capacity?*

The channel resulting from the use of parallel channels with inputs $(1,0)$ (photon only on the first channel) and $(0,1)$ (photon only on the second channel) is illustrated in Figure 5. Note that the input sequences $(X_1, X_2) = (0,0)$

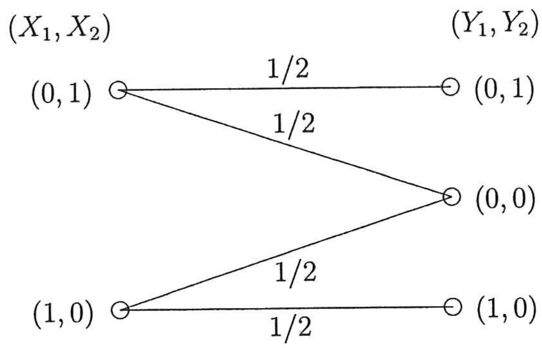


Figure 5: Resulting combined channel for Question 1(b)

and $(X_1, X_2) = (1,1)$ are not included because our transmission strategy never transmits them, whereas the output sequence $(Y_1, Y_2) = (1,1)$ is not included because it occurs with probability 0 given that we never transmit $(X_1, X_2) = (1,1)$ and that the channels are such that we can never have an output 1 for an input 0.

The resulting channel is a binary erasure channel (BEC) with erasure probability $p = 1/2$. Its capacity is

$$C = 1 - p = \frac{1}{2}.$$

Assessor's comment:

A popular question that proved difficult for a number of the candidates. Part (a) was well-answered by most candidates, but the remaining 3 parts gave more problems. In part (d), a number of the candidates could not determine the equivalent binary erasure channel.

Question 2

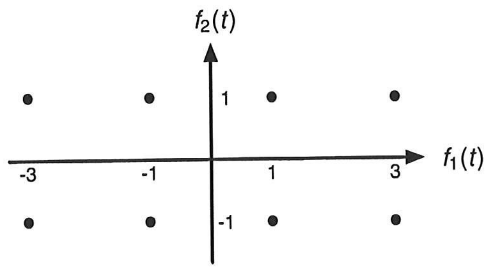


Figure 6: Signal constellation for Question 2.

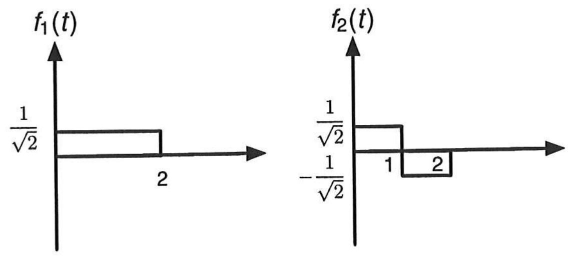


Figure 7: Signals for Question 2.

(a) *Consider the signal constellation shown in Fig. 6. What is the dimension of the signal space?*

The dimension is 2 because there are 2 basis functions.

(b) *Write down the vector representation as a function of $f_1(t)$ and $f_2(t)$.*

The vectors are $(3,1),(3,-1),(1,1),(1,-1),(-1,1),(-1,-1),(-3,1),(-3,-1)$.

(c) *What is the minimum distance of the signal constellation?*

The minimum distance is the distance between $(3,1)$ and $(3,-1)$ which is 2.

(d) *Find the energy of each signal vector and the corresponding average energy.*

The energy of a signal vector $x = (x_1, x_2)$ is $x_1^2 + x_2^2$. Accordingly, the energies of the 8 signal vectors are $10, 10, 2, 2, 2, 2, 10$ and 10. The average energy is $(4 \times 10 + 4 \times 2)/8 = 6$.

(e) *Given the signals $f_1(t)$ and $f_2(t)$ shown in Figure 7, verify they are a suitable basis for the signal space.*

We have

$$\int_{-\infty}^{\infty} f_1^2(t)dt = 2\left(\frac{1}{\sqrt{2}}\right)^2 = 1$$
$$\int_{-\infty}^{\infty} f_2^2(t)dt = 1 \cdot \left(\frac{1}{\sqrt{2}}\right)^2 + 1 \cdot \left(-\frac{1}{\sqrt{2}}\right)^2 = 1$$

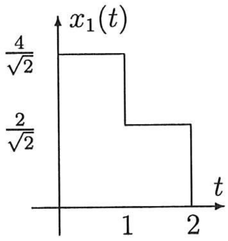
so both signals have norm 1. We also have

$$\int_{-\infty}^{\infty} f_1(t)f_2(t)dt = 1 \cdot \left(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right) + 1 \cdot \left(\frac{1}{\sqrt{2}}\frac{-1}{\sqrt{2}}\right) = 0$$

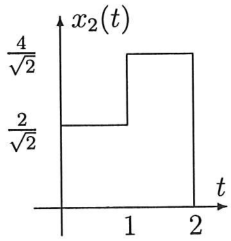
so the signals are orthogonal. The signals therefore form a suitable orthonormal basis for the signal space.

(f) *Sketch the 8 signals $x_1(t), \ldots, x_8(t)$ corresponding to the signal constellation in Fig. 6 using the basis functions $f_1(t)$ and $f_2(t)$ shown in Figure 7.*

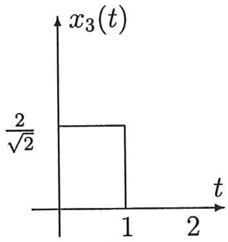
$$x_1(t) = 3f_1(t) + f_2(t)$$



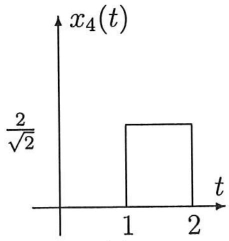
$$x_2(t) = 3f_1(t) - f_2(t)$$



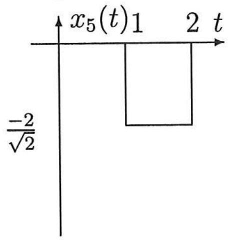
$$x_3(t) = f_1(t) + f_2(t)$$



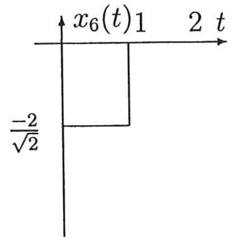
$$x_4(t) = f_1(t) - f_2(t)$$



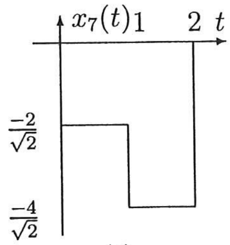
$$x_5(t) = -f_1(t) + f_2(t)$$



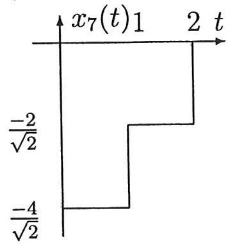
$$x_6(t) = -f_1(t) - f_2(t)$$



$$x_7(t) = -3f_1(t) + f_2(t)$$



$$x_8(t) = -3f_1(t) - f_2(t)$$



Assessor's comment:

A popular and straightforward question that presented little difficulty to most candidates. Surprisingly, some candidates incorrectly calculated the average energy of the constellation.

Question 3

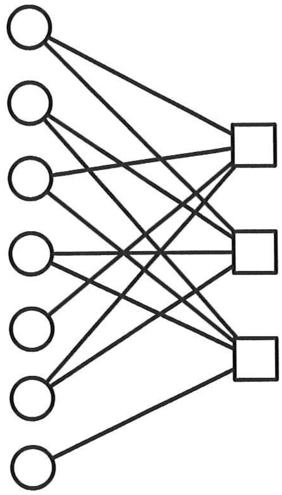


Figure 8: Factor graph for Question 3.

(a) *Consider the code whose factor graph representation is given in Figure 8. Write down the parity-check equations and the corresponding parity-check matrix. What is the code rate?*

The three parity check equations are

$$x_1 + x_3 + x_5 + x_6 = 0$$
$$x_1 + x_2 + x_4 + x_6 = 0$$
$$x_2 + x_3 + x_4 + x_7 = 0$$

The corresponding parity-check matrix is

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

The dimension of the dual code being 3 (the number of linearly independent rows in the parity-check matrix), the dimension of the code is $7 - 3 = 4$. Therefore, the code rate is

$$R = \frac{k}{n} = \frac{4}{7} = 0.5714$$

where k is the code dimension and n is the codeword length.

(b) *Write down the variable and check-node degree distributions (edge perspective) of the code, interpreted as a low-density parity-check code, i.e., λ_i and ρ_i.*

The number of edges connected to variable nodes of degrees 1 and 2 is 2 and 10, respectively. Therefore, the left degree distribution, expressed as a polynomial, is

$$\lambda(x) = \lambda_1 x^0 + \lambda_2 x^1 = \frac{2}{12} + \frac{10}{12}x,$$

or equivalently expressed as a vector $(\lambda_1, \lambda_2) = (1/6, 5/6)$. The number of edges connected to check nodes of degrees 4 is 12. Therefore, the right degree distribution, expressed as a polynomial, is

$$\rho(x) = \sum_i \rho_i x^{i-1} = x^3,$$

or equivalently expressed as a vector $\rho = (0, 0, 0, 1)$.

(c) *Find a systematic generator matrix.*

We first bring the parity-check matrix into systematic form by adding the second row to the first

$$H_s = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

We then transform the systematic parity-check matrix $H_s = [P^T|I]$ into a systematic generator matrix

$$G_s = [I|P] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(d) *The code is used for transmission over a binary erasure channel. The received word is $?, ?, 1, 1, ?, 0, 0$, where ? denotes the erasure symbol. Can the iterative decoder successfully decode? What is the decoded codeword and what are the decoded information bits?*

We apply the last parity-check equation first

$$x_2 + x_3 + x_4 + x_7 = x_2 + 1 + 1 + 0 = 0$$

and obtain $x_2 = 0$.

We then apply the second parity-check equation

$$x_1 + x_2 + x_4 + x_6 = x_1 + 0 + 1 + 0 = 0$$

and obtain $x_1 = 1$. Finally, we apply the first parity-check equation

$$x_1 + x_3 + x_5 + x_6 = 1 + 1 + x_5 + 0 = 0$$

to obtain $x_5 = 0$ and with it the transmitted codeword $x = [1, 0, 1, 1, 0, 0, 0]$, whose systematic part gives the information symbols $u = [1, 0, 1, 1]$.

Assessor's comment:

A popular and straightforward question that was well-answered by most candidates. Part (c) presented the most difficulty where some candidates were unable to determine a systematic generator matrix for the code.

Question 4

1. *Consider transmission over a fading channel with the multipath intensity profile given in Figure 10 and Doppler spectrum following the Jakes model given by*

$$S_H(\xi) = \begin{cases} \frac{1}{\pi f_m} \frac{1}{\sqrt{1-(\xi/f_m)^2}} & |\xi| \le f_m \\ 0 & |\xi| > f_m \end{cases}$$

where f_m is the maximum Doppler frequency.

   (a) *What is the coherence bandwidth of the channel?*

   The delay spread of the channel is $T_d = 10ms$. Therefore, the coherence bandwidth is $B_c = 1/T_d = 100Hz$.

   (b) *What is the coherence time of the channel if the carrier frequency is $f_c = 10MHz$ and the mobile user is moving at a velocity $v = 10km/h$?*

   The maximum Doppler frequency is $(v = 2.77m/s)$.

$$f_m = vf_c/c = 0.0925Hz$$

   and therefore $T_c = 1/f_m = 10.8s$.

   (c) *Will the channel introduce frequency or time selectivity if codewords of duration $T_x = 20ms$ using signals of bandwidth $B_x = 2$ MHz are employed for transmission.*

   Since $B_x \gg B_c$ the channel is frequency selective. Since $T_x \ll T_c$ the channel is not time selective.

2. *Consider binary transmission over a fast Rayleigh fading channel. Show that the pairwise error probability of an error event at Hamming distance d can be upper bounded by*

$$PEP(d) \le \left(\frac{1}{1+SNR}\right)^d.$$

This is bookwork.

Using that $Q(x) \le e^{-\frac{x^2}{2}}$ we have that

$$\text{PEP}(d|\boldsymbol{h}) = Q\left(\sqrt{2\sum_{i=1}^{d}|H_i|^2\text{SNR}}\right) \le e^{-\sum_{i=1}^{d}|H_i|^2\text{SNR}}$$

Now, the PEP averaged over the fading is given by (defining $\gamma_i = |H_i|^2$)

$$\begin{aligned}
\text{PEP}(d) &= \mathbb{E}[\text{PEP}(d|\boldsymbol{h})] \\
&\leq \mathbb{E}\left[e^{-\sum_{i=1}^{d}|H_i|^2\text{SNR}}\right] \\
&= \int_{\boldsymbol{\gamma}\in\mathbb{R}_+^d} e^{-\sum_{i=1}^{d}\gamma_i\text{SNR}} \underbrace{e^{-\sum_{i=1}^{d}\gamma_i}}_{\text{Fading pdf}} d\boldsymbol{\gamma} \\
&= \prod_{i=1}^{d}\int_0^{\infty} e^{-\gamma_i(1+\text{SNR})} d\gamma_i \\
&= \prod_{i=1}^{d} -\frac{e^{-\gamma_i(1+\text{SNR})}}{1+\text{SNR}}\bigg|_0^{\infty} = \left(\frac{1}{1+\text{SNR}}\right)^d
\end{aligned}$$

3. *Consider the convolutional code shown in Figure 11 followed by an interleaver and an 8-PSK modulator.*

   (a) *What is the code diversity?*

   The code diversity for BICM is the same than that for the binary code, which is the minimum Hamming distance of the code. Now, to find the code diversity of the convolutional code we need to find d_{free}. The trellis section is shown in Fig. 9

   The free distance can be found by tracking the lowest weight path leaving state S_0 and coming back to it. For this code the path is easily seen to be $S_0 \to S_4 \to S_6 \to S_3 \to S_1 \to S_0$ which gives $d_{\text{free}} = 6$.

   (b) *What is the overall transmission rate?*

   The transmission rate is the rate of the code times the number of bits per symbol of the modulation, i.e., $R = 3 \times 1/2 = 1.5$ bits/channel use.

4. *The output of the modulator is now parsed across 5 transmit antennas. The receiver is equipped with 3 antennas.*

   (a) *What is the transmission rate of the overall scheme?*

   The transmission rate is the rate of the code times the number of bits per symbol of the modulation times the number of transmit antennas, i.e., $R = 3 \times 1/2 \times 5 = 7.5$ bits/channel use.

   (b) *Consider a maximum-likelihood detector. How many metrics does the detector need to compute?*

   The number of metrics is equal to the number of possible candidate vector symbols, which are all possible combinations of 8-PSK symbols

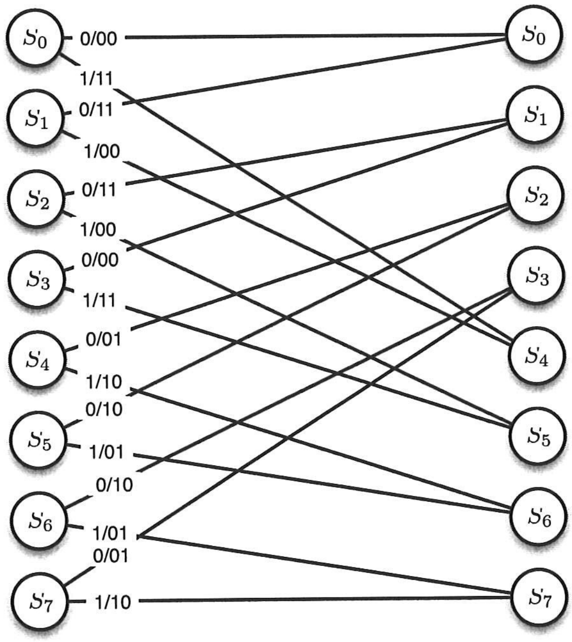


Figure 9: Trellis section for Questions 4.

across the 5 antennas. In this case, the number of metrics is therefore $8^5 = 32768$. At every signaling period, the detector needs to calculate 32768 metrics, compare them and chose the largest.

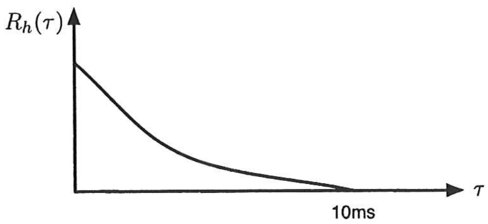


Figure 10: Multipath intensity profile for Question 4.

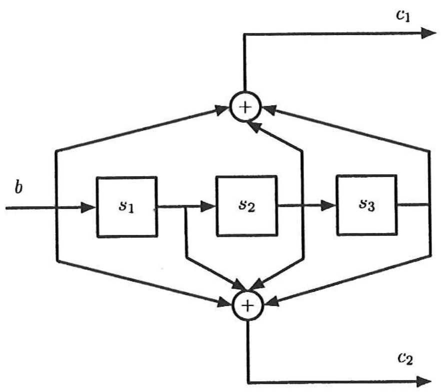


Figure 11: Convolutional code for Question 4.

Assessor's comment:

A very unpopular question. Parts (c) and (d) proved difficult.

4F5 Advanced Wireless Communications, 2010 Numeric Answers

Question 1

(a)

$$\begin{aligned} H(Y) &= 0.8113 \\ H(X,Y) &= \frac{3}{2} \\ H(X|Y) &= 0.6887 \\ H(Y|X) &= 0.5 \\ I(X;Y) &= 0.3113 \end{aligned}$$

(b)

(c) $C = 1$.

(d) The resulting channel is a binary erasure channel (BEC) with erasure probability $p = 1/2$. Its capacity is

$$C = 1 - p = \frac{1}{2}.$$

Question 2

(a) The dimension is 2 because there are 2 basis functions.

(b) The vectors are $(3,1), (3,-1), (1,1), (1,-1), (-1,1), (-1,-1), (-3,1), (-3,-1)$.

(c) The minimum distance is the distance between $(3,1)$ and $(3,-1)$ which is 2.

(d) *Find the energy of each signal vector and the corresponding average energy.*

The energy of a signal vector $x = (x_1, x_2)$ is $x_1^2 + x_2^2$. Accordingly, the energies of the 8 signal vectors are $10, 10, 2, 2, 2, 2, 10$ and 10. The average energy is $(4 \times 10 + 4 \times 2)/8 = 6$.

(e)

(f)

Question 3

(a) The three parity check equations are

$$x_1 + x_3 + x_5 + x_6 = 0$$
$$x_1 + x_2 + x_4 + x_6 = 0$$
$$x_2 + x_3 + x_4 + x_7 = 0$$

The corresponding parity-check matrix is

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

The code rate is

$$R = \frac{k}{n} = \frac{4}{7} = 0.5714$$

where k is the code dimension and n is the codeword length.

(b) Therefore, the left degree distribution, expressed as a polynomial, is

$$\lambda(x) = \lambda_1 x^0 + \lambda_2 x^1 = \frac{2}{12} + \frac{10}{12}x,$$

or equivalently expressed as a vector $(\lambda_1, \lambda_2) = (1/6, 5/6)$. Therefore, the right degree distribution, expressed as a polynomial, is

$$\rho(x) = \sum_i \rho_i x^{i-1} = x^3,$$

or equivalently expressed as a vector $\rho = (0, 0, 0, 1)$.

(c) *Find a systematic generator matrix.*

We first bring the parity-check matrix into systematic form by adding the second row to the first

$$H_s = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

We then transform the systematic parity-check matrix $H_s = [P^T|I]$ into a systematic generator matrix

$$G_s = [I|P] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(d) The transmitted codeword $x = [1, 0, 1, 1, 0, 0, 0]$, whose systematic part gives the information symbols $u = [1, 0, 1, 1]$.

Question 4

1. (a) $B_c = 1/T_d = 100Hz$.

   (b) $T_c = 1/f_m = 10.8s$.

   (c) Frequency selective, not time selective.

2.

3. (a) $d_{\text{free}} = 6$.

   (b) *What is the overall transmission rate?*
   $R = 3 \times 1/2 = 1.5$ bits/channel use.

4. (a) $R = 3 \times 1/2 \times 5 = 7.5$ bits/channel use.

   (b) 32768