PART IIB 2010

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4F6 Solutions 2010.

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4F6: SIGNAL DETECTION AND ESTIMATION

1) The Neyman-Fisher factorization theorem is useful in estimation theory since it helps decide what the "sufficient statistics" are for estimating parameters of a model for data.

The theorem states that if the likelihood $p(x|\theta)$ can be written as

$$p(x|\theta) = g(T(x), \theta)\, h(x)$$

where x is data and θ the parameter of the model, then $T(x)$ is the sufficient statistic for θ.

The theorem can be proved in two ways.

1) For an unbiased, efficient estimator we can write:

$$\frac{\partial}{\partial\theta} \ln p(x|\theta) = k(\theta)\left(\hat{\theta}(x) - \theta\right)$$

$$\therefore \ln p(x|\theta) = \int k(\theta')\left(\hat{\theta}(x) - \theta'\right) d\theta' + \ln h(x).$$

$$\therefore p(x|\theta) = h(x)\, g\left(T(x), \theta\right).$$

The second way uses Bayes theorem.

i) assume $p(x|\theta) = g(T(x),\theta)\, h(x)$

$$\therefore \quad p(\theta|x) = \frac{p(\theta)\, p(x|\theta)}{p(x)}$$

$$= \frac{g(T(x),\theta)\, \cancel{h(x)}\, p(\theta)}{\cancel{h(x)} \int p(\theta)\, g(T(x),\theta)\, d\theta}.$$

$$= p(\theta | T(x))$$

ie $T(x)$ is the only way the data enters into the estimation and is $\therefore$ sufficient.

Q.E.D

The likelihood for the data $\underline{x}$ is

$$p(\underline{x}|\theta) = \prod_{n=0}^{N-1} \exp\left(A(\theta)B(x_n) + C(x_n) + D(\theta)\right)$$

$$= \exp\left\{ A(\theta)\sum B(x_n) + \sum C(x_n) + N D(\theta) \right\}.$$

$$= \exp\left\{ A(\theta)\sum B(x_n) + N D(\theta) \right\} \exp\left\{ \sum C(x_n) \right\}.$$

$\therefore$ using the N-F theorem, the sufficient statistic $T(\underline{x})$ is

$$T(\underline{x}) = \sum_{n=0}^{N-1} B(x_n).$$

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a) Gaussian case (+ likewise for the exponential)

$$p(x|\mu) = \frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}(x-\mu)^2\right) \quad \text{for } \sigma = 1$$

$$= \exp\left\{ x\mu - \tfrac{1}{2}x^2 + \left(-\tfrac{1}{2}\mu^2 + \ln \tfrac{1}{\sqrt{2\pi}}\right) \right\}$$

$$\therefore\ T(\underline{x}) = \sum_{n=0}^{N-1} x_n$$

[illegible]

2) First part is book work (– see notes)

a) The log-likelihood is

$$\ln p(x_0|\theta) = \ln x_0 - \ln\theta - \frac{x_0^2}{2\theta}$$

$$\therefore \frac{\partial p(x_0|\theta)}{\partial\theta} = -\frac{1}{\theta} + \frac{x_0^2}{2\theta^2}$$

and for a ML estimator, this is zero.

$$\therefore -1/\theta + x_0^2/2\theta^2 = 0$$

$$\therefore \hat{\theta} = 1/2\, x_0^2 .$$

b) $$E[\hat{\theta}] = E\left[1/2\, x_0^2\right] = \int_0^\infty \frac{1}{2} x_0^2 \frac{x_0}{\theta} e^{-x_0^2/2\theta} dx = \theta$$

$\therefore$ unbiased

2)

c)

$$p(x_0, x_1, \dots x_{N-1} | \theta) = \prod_{n=0}^{N-1} p(x_n | \theta)$$

$$= \prod_{n=0}^{N-1} \frac{x(n)}{\theta} \exp\left[-\frac{x^2(n)}{2\theta}\right]$$

$$= \frac{1}{\theta^N} \left(\prod_{n=0}^{N-1} x(n)\right) \exp\left(-\frac{1}{2\theta} \sum_{n=0}^{N-1} x^2(n)\right)$$

$$\therefore \ln p(x_0, x_1, \dots x_{N-1} | \theta) = -N \ln \theta + \sum \ln x(n) - \frac{1}{2\theta} \sum x^2(n)$$

$$\therefore \frac{\partial \ln p(x|\theta)}{\partial \theta} = -\frac{N}{\theta} + \frac{1}{2\theta^2} \sum x^2(n) = 0$$

$$\therefore \hat{\theta} = \frac{1}{2N} \sum_{n=0}^{N-1} x^2(n)$$

3) First part is book work – see notes.

b) For the two hypotheses H_0 and H_1, the likelihoods are

$$p(\underline{y}|H_0) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp\left(-\frac{1}{2\sigma^2}\underline{y}^T\underline{y}\right)$$

&

$$p(\underline{y}|H_1) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp \frac{-1}{2\sigma^2}(\underline{y}-\underline{s})^T(\underline{y}-\underline{s})$$

The likelihood ratio test is

$$L(\underline{y}) = \frac{p(\underline{y}|H_1)}{p(\underline{y}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} k$$

where k depends upon the detection criteria used i.e. MAP, Bayes or Neyman-Pearson.

3 cont)

Therefore

$$L(\underline{y}) = \exp \frac{1}{2\sigma^2}\left(2\underline{y}^T\underline{s} - \underline{s}^T\underline{s}\right)$$

taking logs,

$$\therefore \quad \underline{y}^T\underline{s} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}\underline{s}^T\underline{s} + \sigma^2 \ln k.$$

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c) for the case of coloured noise with covariance matrix C the detector becomes

$$y^T C^{-1} s \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2} s^T C^{-1} s + \ln k.$$

using decomposition methods for C ie (cholesky etc

we get $$y^{T'} s' \underset{H_0}{\overset{H_1}{\gtrless}} \tfrac{1}{2} s^{T'} s' + \ln k.$$

where $y^{T'}$ and s' are prewhitened version of y + s

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4) a) & b) are bookwork – see notes.

c) The general linear model can be written

$$\underline{d} = \underline{G}\,\underline{\theta} + \underline{w}$$

Where $\underline{d}$ is observed data, model parameters are θ and G is a matrix and $\underline{w}$ is additive noise.

For the signal $s(n) = A + Bn$,

the observed data is $\underline{d} = \underline{s} + \underline{w}$

$$\therefore \underline{d} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & \vdots \\ 1 & N-1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} + \underline{w} = \underline{G}\,\underline{\theta} + \underline{w}.$$

Therefore,

$$s(n) = A + Bn$$

can be written as a General linear model.

4 c)

The likelihoods for the two hypotheses H_0 and H_1 are

$$p(\underline{d} \mid H_0) = \frac{1}{(2\pi)^{N/2} |C|^{1/2}} e^{-\frac{1}{2}\underline{d}^T C^{-1} \underline{d}}$$

$$p(\underline{d} \mid H_1) = \frac{1}{(2\pi)^{N/2} |C|^{1/2}} e^{-\frac{1}{2}(\underline{d}-\underline{s})^T C^{-1} (\underline{d}-\underline{s})}$$

The N-P detector is $\dfrac{p(\underline{d} \mid H_1)}{p(\underline{d} \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$

$$\therefore \; d^T C^{-1} s \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2} s^T C^{-1} s + \ln \lambda .$$

or $$\frac{1}{\sigma^2} \sum d(n) (A + Bn) \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$