PART IIB 2010 DR S S SINGH

4F7 Digital filters and spectrum estimation Solutions to the 2010 exam.

Solution to Question 1

Part (a)

(Consult the lecture notes for the echo cancellation diagram.) Let $u(n)$ be the signal out of loudspeaker (speech signal of the far speaker), $\varepsilon(n)$ signal of the near speaker, $H_{\text{room}}[u](n)$, the echo generated by the implies response of the room. Call the signal into the mic $d(n)$, i.e.

$$d(n) = \underbrace{H_{\text{room}}[u](n)}_{\text{echo}} + \underbrace{\varepsilon(n)}_{\text{near speaker speech}}$$

We will filter $\{u(n)\}$ to yield $\underline{h}(n)^{\mathrm{T}}\underline{u}(n)$ where $\underline{u}(n) = (u(n), u(n-1), \ldots, u(n-M+1))^{\mathrm{T}}$ and $\underline{h}(n) = (h_0(n), \ldots, h_{M-1}(n))^{\mathrm{T}}$ is the M-tap RLS filter.

The RLS cost function at time n is

$$J_n(\underline{h}) = \sum_{k=0}^{n} \beta^{n-k}(d(n) - \underline{h}^{\mathrm{T}}\underline{u}(n))^2$$

and the minimizer is the RLS filter $\underline{h}(n)$. $\beta \in [0,1]$ is the forgetting factor and making β closer to 0 will allow the RLS filter to track the changing reverberation properties of the room quicker. There is no tracking when $\beta = 1$.

Part (b)

The nonlinear filter is

$$\begin{aligned} y(n) &= h_0(n)u(n) + h_1(n)u(n-1) \\ &\quad + h_{0,0}(n)u(n)u(n) + h_{0,1}(n)u(n)u(n-1) \\ &= [u(n), u(n-1), u(n)^2, u(n)u(n-1)]\underline{h} \\ &= \underline{u}(n)^T\underline{h} \end{aligned}$$

where $\underline{h}(n) = [h_0(n),\ h_1(n),\ h_{0,0}(n),\ h_{0,1}(n)]^{\mathrm{T}}$.

The LMS algorithm should minimise

$$E\left\{(d(n) - \underline{u}(n)^T\underline{h})^2\right\}.$$

From standard LMS theory, step-size μ should satisfy

$$\mu < 2\lambda_{\max}^{-1}$$

where $\lambda_{\max}^{-1}$ is the maximum eigenvalue of $E\{\underline{u}(n)\underline{u}(n)^T\}$. This will ensure the filter converges in mean. The underlying assumption behind the analysis is the independence assumption:

$$E\{(\underline{u}(n)\underline{u}(n)^T)\,\underline{h}(n)\} \approx E\{\underline{u}(n)\underline{u}(n)^T\}E\{\underline{h}(n)\}$$

The LMS update rule is

$$\underline{h}(n+1) = \underline{h}(n) + \mu(d(n) - \underline{u}(n)^T\underline{h}(n))\underline{u}(n).$$

This update rule is obtained by approximating the ensemble average

$$E\left\{(d(n) - \underline{u}(n)^T\underline{h})\underline{u}(n)\right\}\Big|_{\underline{h}=\underline{h}(n)}$$

with the one term sample average $(d(n) - \underline{u}(n)^T\underline{h}(n))\underline{u}(n)$.

Part (c) Let $R = E\{\underline{u}(n)\underline{u}(n)^T\}$. When third order statistics are zero,

$$R = \begin{bmatrix} R_1 & 0_{2\times 2} \\ 0_{2\times 2} & R_2 \end{bmatrix}$$

where

$$R_1 = E\left\{[u(n), u(n-1)]^{\mathrm{T}}[u(n), u(n-1)]\right\},$$
$$R_2 = E\left\{[u(n)^2, u(n)u(n-1)]^{\mathrm{T}}[u(n)^2, u(n)u(n-1)]\right\}.$$

So the LMS aims to solve for

$$\underline{h}_{\text{opt}} = \begin{bmatrix} R_1^{-1}p_1 \\ R_2^{-1}p_2 \end{bmatrix}$$

where $p_1 = E\left\{[u(n), u(n-1)]^{\mathrm{T}}d(n)\right\}$ and $p_2 = E\left\{[u(n)^2, u(n)u(n-1)]^{\mathrm{T}}d(n)\right\}$. Clearly one could implement two separate LMS algorithms as follows:

$$\underline{h}^{\text{lin}}(n+1) = \underline{h}^{\text{lin}}(n) + \mu_1(d(n) - \underline{u}^{\text{lin}}(n)^T\underline{h}^{\text{lin}}(n))\underline{u}^{\text{lin}}(n),$$
$$\underline{h}^{\text{nlin}}(n+1) = \underline{h}^{\text{nlin}}(n) + \mu_2(d(n) - \underline{u}^{\text{nlin}}(n)^T\underline{h}^{\text{nlin}}(n))\underline{u}^{\text{nlin}}(n)$$

where

$$\underline{u}^{\text{lin}}(n) = [u(n), u(n-1)]^{\mathrm{T}}, \quad \underline{u}^{\text{nlin}}(n) = [u(n)^2, u(n)u(n-1)]^{\mathrm{T}}.$$

The nonlinear filter is then

$$\underline{h}(n) = \begin{bmatrix} \underline{h}^{\text{lin}}(n) \\ \underline{h}^{\text{nlin}}(n) \end{bmatrix}.$$

Clearly now μ_1 and μ_2 should satisfy

$$\mu_i < 0.5/\lambda_{\max}(R_i)$$

where $\lambda_{\max}(R_i)$ is the maximum eigenvalue of R_i.

This scheme is better since we do not slow down convergence by choosing the largest eigenvalue of matrix R as a conservative estimate of the bound for μ_1 and μ_2.

Examiner's comments: The application to echo cancelation was answered very well by most. For part (b) the majority of candidates were able to state the LMS algorithm and gave conditions under which it converged. A significant number did not realise that the non-linear LMS filter could be treated as a standard linear LMS filter with a 4 tap filter and a 4 component input vector sequence. Part (c) was the most challenging. The solution to this part comes by considering the Wiener filter for part (b). The input correlation matrix would be a 4 by 4 matrix with a block diagonal structure. This allows the LMS algorithm in part (b) to be written as two decoupled LMS algorithms.

Solution to Question 2

Part (a)

For the error $e(n) = \widehat{x}(n) - x$. Square it and take the expectation to get:

$$\begin{aligned} e(n)^2 &= K(n)^2 \left(v(n) - e(n-1)\right)^2 + e(n-1)^2 \\ &\quad + 2K(n)\left(v(n) - e(n-1)\right) e(n-1) \end{aligned}$$

$$\begin{aligned} E\left\{e(n)^2\right\} &= K(n)^2 E\left\{v(n)^2 + e(n-1)^2 + 2v(n)e(n-1)\right\} + E\left\{e(n-1)^2\right\} \\ &\quad + 2K(n)E\left\{\left(v(n) - e(n-1)\right)e(n-1)\right\} \\ &= K(n)^2\left(\sigma_v^2 + E\left\{e(n-1)^2\right\}\right) + E\left\{e(n-1)^2\right\} \\ &\quad - 2K(n)E\left\{e(n-1)^2\right\} \end{aligned}$$

The last line follows from the stated assumption on $\{v(n)\}$. Let $\sigma(n)^2 = E\{e(n)^2\}$. Differentiating the right-hand side with respect to $K(n)$ and equating to 0 to solve for $K(n)$ yields:

$$K(n) = \frac{\sigma(n-1)^2}{\sigma_v^2 + \sigma(n-1)^2}.$$

Part (b) At time n=1, assuming $\widehat{x}(0) = 0$, $\widehat{x}(1) = \frac{\sigma_0^2}{\sigma_0^2+\sigma_v^2} y(1)$. At time $n = 2$,

$$\begin{aligned}
\widehat{x}(2) &= \frac{\sigma_0^2}{\sigma_0^2+\sigma_v^2} y(1) + \frac{\sigma_0^2}{2\sigma_0^2+\sigma_v^2}\left(y(2) - \frac{\sigma_0^2}{\sigma_0^2+\sigma_v^2} y(1)\right) \\
&= \frac{\sigma_0^2}{\sigma_0^2+\sigma_v^2} y(1)\left(1 - \frac{\sigma_0^2}{2\sigma_0^2+\sigma_v^2}\right) + \frac{\sigma_0^2}{2\sigma_0^2+\sigma_v^2} y(2) \\
&= \frac{\sigma_0^2}{2\sigma_0^2+\sigma_v^2} y(1) + \frac{\sigma_0^2}{2\sigma_0^2+\sigma_v^2} y(2).
\end{aligned}$$

Assuming at time n-1

$$\widehat{x}(n-1) = \frac{\sigma_0^2}{(n-1)\sigma_0^2+\sigma_v^2} \sum_{i=1}^{n-1} y(i)$$

then,

$$\begin{aligned}
\widehat{x}(n) &= \widehat{x}(n-1)\left(1 - \frac{\sigma_0^2}{n\sigma_0^2+\sigma_v^2}\right) + \frac{\sigma_0^2}{n\sigma_0^2+\sigma_v^2} y(n) \\
&= \widehat{x}(n-1)\frac{(n-1)\sigma_0^2+\sigma_v^2}{n\sigma_0^2+\sigma_v^2} + \frac{\sigma_0^2}{n\sigma_0^2+\sigma_v^2} y(n) \\
&= \frac{\sigma_0^2}{n\sigma_0^2+\sigma_v^2} \sum_{i=1}^{n} y(i).
\end{aligned}$$

Part (c) The variance of the sample mean estimate is

$$E\left\{\left(\frac{1}{n}\sum_{i=1}^{n} y(i)\right)^2\right\} = E\left\{\left(x + \frac{1}{n}\sum_{i=1}^{n} v(i)\right)^2\right\} = \frac{\sigma_v^2}{n} + \sigma_0^2.$$

The variance of the Kalman filter estimate is

$$E\left\{\left(\frac{\sigma_0^2}{n\sigma_0^2+\sigma_v^2}\sum_{i=1}^{n} y(i)\right)^2\right\} = \left(\frac{\sigma_0^2}{n\sigma_0^2+\sigma_v^2}\right)^2 \left(n\sigma_v^2 + n^2\sigma_0^2\right).$$

Since $\frac{\sigma_0^2}{n\sigma_0^2+\sigma_v^2} < \frac{1}{n}$, the result follows.

The calculation of the mean squared error is also acceptable. For the Kalman filter, $E\left\{(\widehat{x}(n) - x)^2\right\} = (n + \frac{\sigma_v^2}{\sigma_0^2})^{-1}\sigma_v^2$. For the sample mean estimator, the value is $n^{-1}\sigma_v^2$.

Examiner's comments: Part (a) was answered well by most and so was part (b). Part (c) was disappointing. Not many used the fact that the variance of the sum of independent random variables was the sum of the variance of the corresponding variables despite it being stressed in lectures. This meant calculations were significantly complicated and largely incomplete.

Solution to Question 3

Part (a)

When $\gamma_k = k^{-1}$, the estimate $\widehat{S}^{(k)}$ is averaging periodograms computed over subsequences of data of size N. If the signal being estimated is stationary then variance of $\widehat{S}^{(k)}$ will tend to 0. When $\gamma_k = \alpha$ for all k, then variance of $\widehat{S}^{(k)}$ will not tend to 0. This approach is more suited to estimating a power spectrum that is changing over time. The larger α is, the quicker the estimate can track the changes in the spectral content of the signal. However the variance will also increase. (The opposite effect is observed as α is made smaller.)

The bias of the estimate decreases as N increases. The frequency resolution, or the ability to discriminate two closely spaced frequencies in the power spectrum estimate, improves as N increases. One could use a modified periodogram instead to alter the frequency resolution and spectral leakage properties of $\widehat{S}^{(k)}$.

Part (b)

Use the following results for uncorrelated random variables

$$\text{var}\left\{\sum_k a_k X_k\right\} = \sum_k a_k^2 \text{var}\left\{X_k\right\}.$$

Let $P^{(k)}(e^{j\omega}) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x_n^{(k)} e^{-jn\omega}\right|^2$. Thus we have

$$\operatorname{var}\left\{\widehat{S}^{(k)}\right\} = (1-\gamma_k)^2 \operatorname{var}\left\{\widehat{S}^{(k-1)}\right\} + \gamma_k^2 \operatorname{var}\left\{P^{(k)}\right\}.$$

It is given that $\widehat{S}^{(0)} = 0$. For $\gamma_k = \alpha$, this is a geometric series:

$$\begin{aligned}\operatorname{var}\left\{\widehat{S}^{(k)}\right\} &= \sum_{m=1}^{k} \underbrace{\left[(1-\alpha)^2\right]}_{r}^{k-m} \underbrace{\alpha^2 \operatorname{var}\left\{P^{(m)}\right\}}_{a} \\ &= \sum_{m=1}^{k} r^{k-m} a\end{aligned}$$

For $\gamma_k = 1/k$, $\widehat{S}^{(k)}$ is a sample average of $P^{(1)}$ to $P^{(k)}$ and thus

$$\operatorname{var}\left\{\widehat{S}^{(k)}\right\} \sum_{m=1}^{k} \frac{1}{k^2} \operatorname{var}\left\{P^{(m)}\right\} = \operatorname{var}\left\{P^{(1)}\right\}/k.$$

(Note that $\operatorname{var}\left\{P^{(m)}\right\} = v$ for some constant since each subsequence has the same variance and mean. This constant will change with frequency ω though.)

When input sequence is a Gaussian process with zero mean and variance σ^2 than we know from the lecture notes that the variance and mean of the periodograms can be calculated exactly to give

$$\begin{aligned}\text{mean} &= \sigma^4 \\ \text{variance} &= \sigma^4 \left(1 + \left\{\frac{\sin(N\omega)}{N\sin(\omega)}\right\}^2\right)\end{aligned}$$

and the variance tends σ^4 as N tend to infinity.

When $\gamma_k = k^{-1}$, $v^{-1}\operatorname{var}\left\{\widehat{S}^{(k)}\right\} = k^{-1}$. Limiting variance is 0. When $\gamma_k = \alpha$, we have a geometric series and $v^{-1}\operatorname{var}\left\{\widehat{S}^{(k)}\right\}$ will converge to $\frac{\alpha^2}{1-(1-\alpha)^2} = \frac{\alpha}{2-\alpha}$.

Part c) Need to know that the power spectrum of $\{x_n\}$ is the sum of two delta functions centered at ω_1 and ω_2, with amplitudes $A^2\pi/2$ and $B^2\pi/2$ respectively, plus a constant noise floor. The expected value of the periodogram will be $0.5/\pi$ times the convolution of the DTFT of the rectangular window $W(e^{j\omega})$ with the power spectrum of $\{x_n\}$. The convolution will shift the main lobe of $W(e^{j\omega})$ to be centered at ω_1 and ω_2. Also, $0.5/\pi \times W(e^{j\omega}) *$constant is zero since the periodogram is unbiased for white noise.

ω_1 and ω_2 can be resolved provided

$$\omega_2 - \omega_1 \geq 2 \times \frac{1.78(2\pi)}{M}$$

where $M = 2N - 1$. Solving for M gives

$$M \geq 2 \times \frac{1.78(2\pi)}{\omega_2 - \omega_1} \geq 2 \times \frac{1.78(2\pi)}{0.05\pi} = 142.4$$

Thus need $N \geq 143.4/2 = 71.7$.

Part (d) Once the desired frequency resolution has been met, it is wise to use the available data to reduce the variance of the estimator, i.e. form more periodograms to average over.

Examiner's comments: Part (a) was answered well with almost all correctly identifying this as a scheme to average periodograms. Not all knew that the constant step-size meant the estimator could track. A discussion of the variance of the two methods was missing. Part (b) proved difficult for many. This was primarily because candidates did not use the fact that the periodograms were uncorrelated as mentioned in the question. The main weakness was not recognizing that the variance of the sum of uncorrelated random variables was the sum of the variances, despite it being stressed in lectures. Part (c) should have been very straightforward and an easy mark earner for all but unfortunately it was only the case for some. This question had been addressed in lectures and the examples paper.

Solution to Question 4

Part (a) Periodogram is a non-parametric spectrum estimation technique while the maximum likelihood (ML) assumes a parametric model, e.g. the AR model.

The periodogram will have higher variance because the autocorellation is estimated for all lags (the final lag is number of data points minus one).

(Techniques to improve the periodogram estimate could have been discussed as well.)

The ML estimate of the power spectrum is obtained by using ML estimate of the model parameters in the power spectrum expression of the model.

Because the model order is typically smaller than the size of the data set, the variance of the estimate will be less.

In the simplest case, the ML method will assume something about the data before time zero, which is not ideal and can result in a bias when the size of the data set is comparable to the model order.

For the complex signal $Ae^{j\omega_0 n}$ observed in complex Gaussian noise, the ML estimate of ω_0 coincides with the maximum of the periodogram power spectrum estimate.

Part (b-i)

For the MA model, $c_0 = R_{XX}[0]$, $c_1 = R_{XX}[1]$

Now factorise the polynomial $\sum_{r=-Q}^{Q} \widehat{R}_{XX}[r] z^{-r} = z + 2 + z^{-1} = (1 + z)(1 + z^{-1})$

Take the root $z = -1$ (although, strictly speaking, we should have a root within the unit circle.)

Now write $B(z) = g(1 - z^{-1} n_1)$ where n_1 is the identified root.

The constant g is

$$\sqrt{\frac{\widehat{R}_{XX}[0]}{1 + (-n_1)^2}} = 1$$

The MA model parameters are: $b_0 = g = 1$, $b_1 = g \times n_1 = 1$.

This is the spectral factorisation technique.

Part (b-ii) The signal model is $x_n = -a_1 x_{n-1} + b_0 w_n + b_1 w_{n-1}$. From the given equations, $c_0 = b_0 h_0 + b_1 h_1$, $c_1 = b_1 h_0$ and

$$\begin{bmatrix} R_{XX}[0] + a_1 R_{XX}[-1] \\ R_{XX}[1] + a_1 R_{XX}[0] \\ R_{XX}[2] + a_1 R_{XX}[1] \end{bmatrix} = \begin{bmatrix} b_0 h_0 + b_1 h_1 \\ b_1 h_0 \\ 0 \end{bmatrix}$$

Use the third equation to solve for a_1 to get $a_1 = -\widehat{R}_{XX}[2] / \widehat{R}_{XX}[1] = 1$.

Note now we have 2 equations and 4 unknowns. So the trick is to observe that

$$y_n = x_n + a_1 x_{n-1}$$

is a ARMA(0,1) model and use the previous method to solve for the coefficients. We need $R_{YY}[0]$ and $R_{YY}[1]$.

$$\begin{aligned} R_{YY}[0] &= E(y_n y_n) \\ &= E\left(x_n x_n + a_1^2 x_{n-1} x_{n-1} + 2 a_1 x_n x_{n-1}\right) \\ &= 2R_{XX}[0] + 2R_{XX}[1] \end{aligned}$$

Similarly

$$\begin{aligned} R_{YY}[1] &= E(y_{n+1}y_n) \\ &= E\left(x_{n+1}x_n + a_1^2 x_n x_{n-1} + a_1 x_{n+1} x_{n-1} + a_1 x_n x_n\right) \\ &= R_{XX}[0] + 2R_{XX}[1] + R_{XX}[2] \end{aligned}$$

So $\widehat{R}_{YY}[0] = 6$ and $\widehat{R}_{YY}[1] = 3$

Now factorise the polynomial $\sum_{r=-Q}^{Q} \widehat{R}_{YY}[r] z^{-r} = 3z + 6 + 3z^{-1} = 3(1 + z)(1 + z^{-1})$

Take the root $z = -1$. Now write $B(z) = g(1 - z^{-1}n_1)$ where n_1 is the identified root.

The constant g is

$$\sqrt{\frac{\widehat{R}_{YY}[0]}{1 + (-n_1)^2}} = \sqrt{3}$$

The MA model parameters are: $b_0 = g = \sqrt{3}$, $b_1 = g \times n_1 = -\sqrt{3}$.

Part(c) The ARMA(0,1) will give an autocorellation value of 0 for lags larger than 1 which is inconsistent with the data. So the ARMA(1,1) model is preferable.

Examiner's comments: Part (a) was answered well and many earned high marks. There was ample to discuss on the two methods. Part (b.i) was straightforward as a similar question was discussed in the lectures and examples papers. I was disappointed that not all had earned high marks for this question. Part (b.ii) was challenging for all but the best. This question should have been answered by using the solution to part (b.i.). Specifically, this model could be written as a ARMA(0,1) model by grouping the x component of the model on the left-hand side. This should have earned many good marks.